Expected Shapley Value is Shapley Value for Expected Utility Game

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Shapley-like scores

- N: finite set of players / variables
- $v : 2^N \to \mathbb{Q}$: set function (general game/utility function)
- $c : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$: coefficient function (assumed PTIME computable on unary input)

$$\mathrm{Score}_c(v, N, i) \stackrel{\text{def}}{=} \sum_{C \subseteq N \setminus \{i\}} c(|N|, |C|) \left[v(C \cup \{i\}) - v(C) \right].$$

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Example

- $c_{\mathrm{Shapley}}(k, \ell) \stackrel{\text{def}}{=} \frac{\ell!(k-\ell-1)!}{k!} = \binom{k-1}{\ell}^{-1} k^{-1}$: **Shapley value** [Shapley et al., 1953]
- $c_{\mathrm{Banzhaf}}(k, \ell) \stackrel{\text{def}}{=} 1$: **Banzhaf value** [Banzhaf III, 1964]
- $c_{\mathrm{PB}}(k, \ell) \stackrel{\text{def}}{=} 2^{-k+1}$: **Normalized–Banzhaf power** [Kirsch and Langner, 2010]

What is Shapley Value?

- Fair attribution in cooperative games via marginal contributions.
- Classic axioms: **efficiency**, **symmetry**, **linearity**, **null-player**.
- Widely used for feature/data valuation and explanation.

Shapley Value in Databases

- Think of a database as a collection of facts (tuples).
- We ask a Boolean query (a yes/no condition on the data).
- For each subset of tuples, the query may be true (1) or false (0).
- This defines a set function: which coalitions of tuples make the query true.
- The Shapley value then measures how much each tuple contributes on average to the query being satisfied.

Example: Shapley values in a Database

Area

ID	Region	Area	Prov.
01	Valparaiso	16,000	A
02	Atacama	75,000	B
03	Metropolitan	15,000	C
04	Maule	30,000	D

Density

ID	Pop_den	Prov.
01	110	a
02	4	b
03	461	c
04	34	d

```
SELECT DISTINCT 1
FROM Area a JOIN Density d ON a.ID = d.ID
WHERE Area < 20000 AND Pop_den >= 100;
```

Provenance: $\varphi_{ex} = (A \wedge a) \vee (C \wedge c)$

Contributions (Shapley scores):

$x \in V$	Score(φ_{ex}, V, x)
A	0.25
a	0.25
C	0.25
c	0.25
	1.0

Why Expected Values in a Probabilistic Setup?

- Player/tuple participation is stochastic (possible worlds).
- We must measure contribution *in expectation* across realizations.
- Leads to the EShap (Expected Shapley Value).

Definition of EShap

- Suppose we have a probabilistic game: each player $x \in N$ is present independently with some probability.
- For any probabilistic game $\mathcal{G} = (N, (p_i)_{i \in N})$, any utility function v over N, and any player $i \in N$, we have, the Expected Shapley Value:

$$\mathrm{EShap}(v, \mathcal{G}, i) \stackrel{\text{def}}{=} \sum_{Z \subseteq N, i \in Z} \left(\Pi_N(Z) \times \mathrm{Score}_{c_{\mathbf{Shapley}}}(v, Z, i) \right)$$

$$= \sum_{Z \subseteq N, i \in Z} \left(\Pi_N(Z) \times \sum_{C \subseteq Z \setminus \{i\}} c_{\mathbf{Shapley}}(|Z|, |C|) \times [v(C \cup \{i\}) - v(C)] \right)$$

$$= p_i \times \sum_{C \subseteq N \setminus \{i\}} (v(C \cup \{i\}) - v(C)) \left(\sum_{C \subseteq Z \subseteq N \setminus \{i\}} c_{\mathbf{Shapley}}(|Z| + 1, |C|) \Pi_{N \setminus \{i\}}(Z) \right)$$

Where, $\Pi_Z(C) \stackrel{\text{def}}{=} \prod_{j \in C} p_j \prod_{j \in Z \setminus C} (1 - p_j)$

Definition of ShapE

- For any probabilistic game $\mathcal{G} = (N, (p_i)_{i \in N})$ and utility function v:
- Define the expected utility game with utility

$$\mathrm{EV}(v)(S) \overset{\text{def}}{=} \sum_{T \subseteq N} \Pi_N(T) \cdot v(S \cap T),$$

  where $\Pi_N(T)$ is the probability that exactly the set T of players is present.
- Then the Shapley value of the expected utility game is:

$$\mathrm{ShapE}(v, \mathcal{G}, i) \overset{\text{def}}{=} \sum_{Z \subseteq N \setminus \{i\}} c_{\mathrm{Shapley}}(|N|, |Z|) \times (\mathbb{E}_{\mathcal{G}}(v(Z \cup \{i\})) - \mathbb{E}_{\mathcal{G}}(v(Z)))$$

$$= \sum_{Z \subseteq N \setminus \{i\}} c_{\mathrm{Shapley}}(|N|, |Z|) \left(\sum_{C_1 \subseteq Z \cup \{i\}} \Pi_{Z \cup \{i\}}(C_1) v(C_1) - \sum_{C_2 \subseteq Z} \Pi_Z(C_2) v(C_2) \right)$$

$$= p_i \times \sum_{C \subseteq N \setminus \{i\}} (v(C \cup \{i\}) - v(C)) \left(\sum_{C \subseteq Z \subseteq N \setminus \{i\}} c_{\mathrm{Shapley}}(|N|, |Z|) \times \Pi_Z(C) \right)$$

Example with Probabilities: Expected Shapley Scores

Now assume each tuple appears independently with a probability:

Area (with probs)

ID	Region	Area	Prob.	Prov.
01	Valparaiso	16,000	0.4	A
02	Atacama	75,000	0.3	B
03	Metropolitan	15,000	0.6	C
04	Maule	30,000	0.8	D

Density (with probs)

ID	Pop_den	Prob.	Prov.
01	110	0.5	a
02	4	0.2	b
03	461	0.8	c
04	34	0.9	d

Query probability:

$$\Pr(\varphi_{ex}) = 1 - (1 - p_A \times p_a)(1 - p_C \times p_c) = 0.584$$

Contributions (Expected Shapley):

$x \in V$	$\text{Score}(\varphi_{ex}, V, x)$	$\text{EScore}(\varphi_{ex}, x)$
A	0.25	0.076
a	0.25	0.076
C	0.25	0.216
c	0.25	0.216
	1.0	0.584

EShap = ShapE

- EShap: Shapley value computed over realizations, then averaged.
- ShapE: Shapley value of the expected-utility game.

$$\boxed{\mathrm{EShap}(v, \mathcal{G}, i) = \mathrm{ShapE}(v, \mathcal{G}, i)}$$

Algorithmic implication: tractability

- In our previous work [Karmakar et al., 2024], we proved tractability for the Expected Shapley value (EShap).
- **Key reduction:** For any tractable coefficient c, $\mathrm{EScore}_c(F) \leqslant_{\mathrm{P}} \mathrm{EV}(F)$. Thus, if $\mathrm{EV}(F) \in \mathrm{P}$, all EScore_c are in P (e.g., decision trees, OBDDs, d-D circuits, bounded treewidth).
- **Shapley:** For any reasonable class F (with $v(\emptyset)$ in P), $\mathrm{EScore}_{c_{\mathbf{Shapley}}}(F) \equiv_{\mathrm{P}} \mathrm{EV}(F)$.
- **Concrete PT on d-D circuits:** Polytime for EScore_c [Karmakar et al., 2024]; for c_{Banzhaf}, $O(|C| \cdot |V|)$ on tight d-D circuits.
- **This work:** Since EShap = ShapE, all results extend to the Shapley value of the expected-utility game (ShapE).

$$\mathrm{EShap}(v, \mathcal{G}, i) = \mathrm{ShapE}(v, \mathcal{G}, i) \;\Rightarrow\; \big(\mathrm{EShap} \in \mathrm{P} \text{ on the DB class}\big) \;\Rightarrow\; \big(\mathrm{ShapE} \in \mathrm{P}\big).$$

Key Statements of the Proof

Compare coefficients of $(v(C \cup \{i\}) - v(C))$ in:

$$\sum_{C \subseteq Z \subseteq N \setminus \{i\}} c_{\text{Shapley}}(|Z|+1, |C|)\, \Pi_{N \setminus \{i\}}(Z) \stackrel{?}{=} \sum_{C \subseteq Z \subseteq N \setminus \{i\}} c_{\text{Shapley}}(|N|, |Z|)\, \Pi_Z(C)$$

(Coefficient comparison: LHS = RHS)

Example: 2 players

Setup. $N = \{1, 2\}$ with independent availability $p_1 = 0.4$, $p_2 = 0.7$.
Value function: $v(\emptyset) = 0$, $v(\{1\}) = 1$, $v(\{2\}) = 2$, and $v(\{1,2\}) = 5$.

EShap for player 1.

$$\text{EShap}(v, G, 1) = \sum_{Z \subseteq N, 1 \in Z} \Pi_N(Z) \sum_{C \subseteq Z \setminus 1} c_{\text{Shapley}}(|Z|, |C|) \cdot \Big(v(C \cup \{1\}) - v(C) \Big)$$

Two possible values for Z: $\{1\}$ and $\{1, 2\}$:

- $Z = \{1\}$: $\Pi_N(\{1\}) \sum_{C \subseteq \{\emptyset\}} c_{\text{Shapley}}(1, |C|) \cdot \Big(v(C \cup \{1\}) - v(C) \Big)$
  $= p_1(1 - p_2) \frac{0!(1-0-1)!}{1!}(1 - 0) = 0.4 \times 0.3 \times 1 \times 1 = 0.12$
- $Z = \{1, 2\}$: $\Pi_N(\{1,2\}) \sum_{C \subseteq \{2\}} c_{\text{Shapley}}(2, |C|) \cdot \Big(v(C \cup \{1\}) - v(C) \Big)$
  $= p_1 p_2 \Big(c_{\text{Shapley}}(2, 0) \cdot (v(\{1\} - v\{\emptyset\}) + c_{\text{Shapley}}(2, 1) \cdot (v(\{1, 2\} - v\{2\}) \Big)$
  $= 0.4 \times 0.7 \Big(\frac{0!(2-0-1)!}{2!} \times 1 + \frac{1!(2-1-1)!}{2!} \times 3 \Big) = 0.28 \times 2 = 0.56$

Which finally gives $\text{EShap}(v, G, 1) = 0.12 + 0.56 = \boxed{0.68}$.

Example: 2 players

ShapE for player 1.

Define the expected utility $\bar{v}(S) = \mathbb{E}_Z[v(S \cap Z)]$. Recall that:
$\text{ShapE}(v, G, 1) = \sum_{Z \subseteq N \setminus 1} c_{\text{Shapley}}(|N|, |Z|)\Big(\bar{v}(Z \cup \{1\}) - \bar{v}(Z)\Big)$

$$\bar{v}(\{1\}) = p_1 \cdot v(\{1\}) = 0.4, \qquad \bar{v}(\{2\}) = p_2 \cdot v(\{2\}) = 1.4,$$

$$\bar{v}(\{1,2\}) = p_1(1 - p_2) \cdot v(\{1\}) + (1 - p_1)p_2 \cdot v(\{2\}) + p1p2 \cdot v(\{1,2\})$$

$$= 0.4 \cdot 0.3 \cdot 1 + 0.6 \cdot 0.7 \cdot 2 + 0.4 \cdot 0.7 \cdot 5 = 0.12 + 0.84 + 1.40 = 2.36.$$

Shapley on $\bar{v}$ (two-player formula):

$$\text{ShapE}(v, G, 1) = \tfrac{1}{2}\big(\bar{v}(\{1\}) - \bar{v}(\emptyset)\big) + \tfrac{1}{2}\big(\bar{v}(\{1,2\}) - \bar{v}(\{2\})\big)$$

$$= \tfrac{1}{2}(0.4) + \tfrac{1}{2}(2.36 - 1.4) = 0.20 + 0.48 = \boxed{0.68}.$$

Thus: $\text{EShap}(v, G, 1) = \text{ShapE}(v, G, 1)$.

What About Banzhaf?

- Replacing Shapley with *unnormalized* Banzhaf breaks the equality: EBanz $\neq$ BanzE.

Normalized Banzhaf Works

- With Normalized Banzhaf (coeff. 2^{-k+1}), equality is restored via a different argument.
- General family a^{-k+m} preserves equality only for $a = 2$.

Non-Independent Players

- The proof relies on independence; with correlations it does not directly extend.
- However, any correlated game can be mapped to an *independent* one by modifying the utility, preserving equality but typically losing tractability.
- Open direction: efficient handling and characterization under dependencies.

Conclusion

- EShap $=$ ShapE gives robust attribution under uncertainty.
- Banzhaf fails unless normalized; normalized Banzhaf works.
- Practical impact for probabilistic DBs and stochastic games.

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