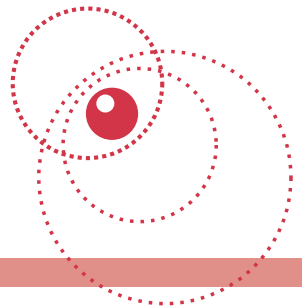

Approximations Algorithms

(for Database Researchers)

Masses de données distribuées, 9 June 2016

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Influence Maximization. Given a social network with influence probabilities on edges, find the set of nodes to target to maximize the impact of a marketing campaign.

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Probabilistic Query Evaluation. Compute the probability of a fixed SQL query over a database whose tuples are annotated with probabilities.

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Polynomial-time	NP-hard	#P-hard
Max. Matching	Set Cover	Coloring Counting
Nav. XPath Counting	Vertex Cover	SQL Match Counting
	Incons. Data Repair	Prob. Query Evaluation
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  - Find randomized algorithms that provide a guaranteed approximation with high probability

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Technical note: most definitions of NP-hardness for decision problems are a bit stricter because they are based on Karp many-one reductions. Important if you want to distinguish, e.g., NP-hardness vs coNP-hardness. For optimization/computation problems, it is irrelevant, so I use simpler Turing reductions.

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- But Y reduces to Z: use Z to count, and return "yes" iff the count is > 0.
- Therefore Y reduces to X, and thus X is NP-hard.

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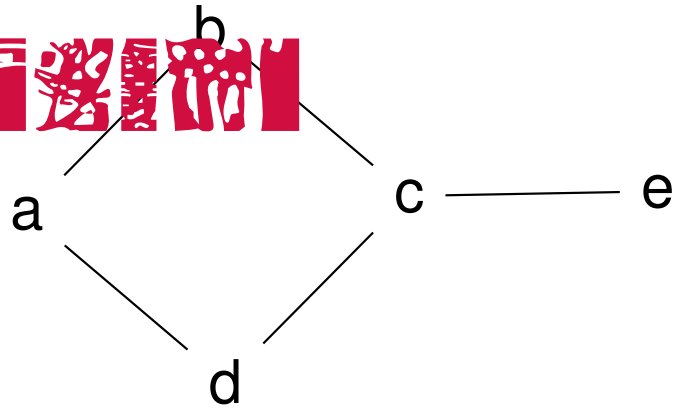
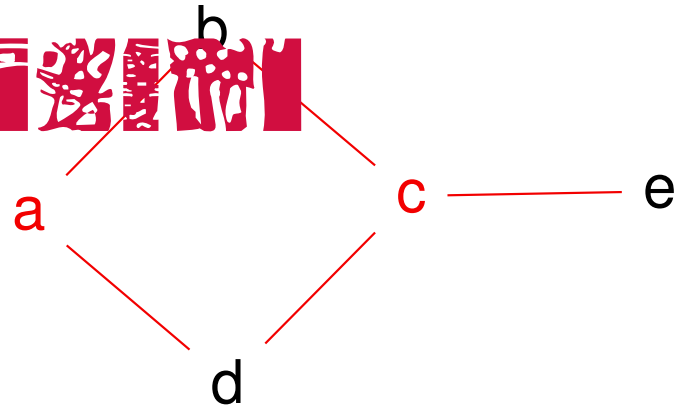
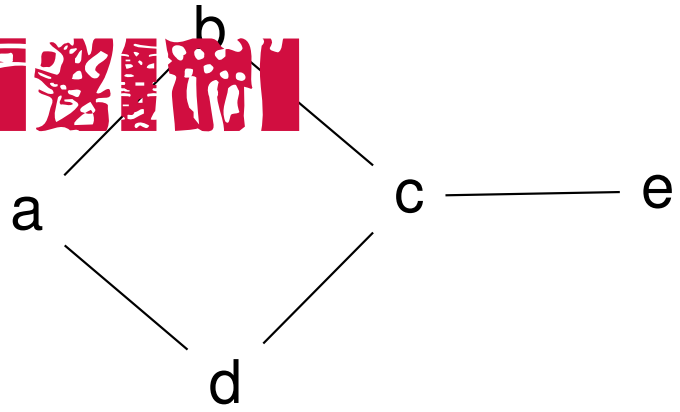
- Additive approximation is thus rarely used, “approximation algorithm” usually means multiplicative
- APX: class of optimization problems that have a polynomial-time multiplicative approximation algorithm with constant φ





Optimal: $\{a, c\}$, size 2



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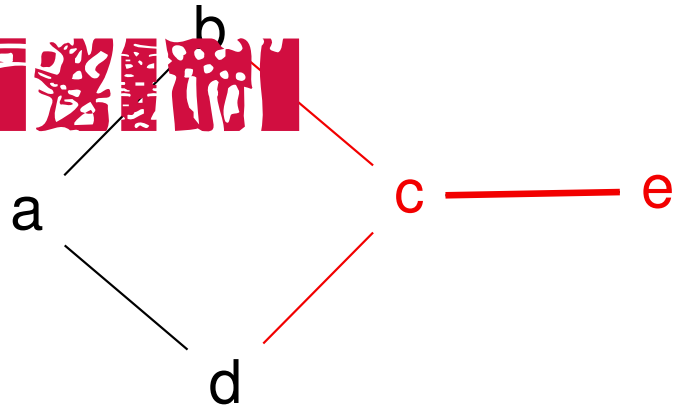

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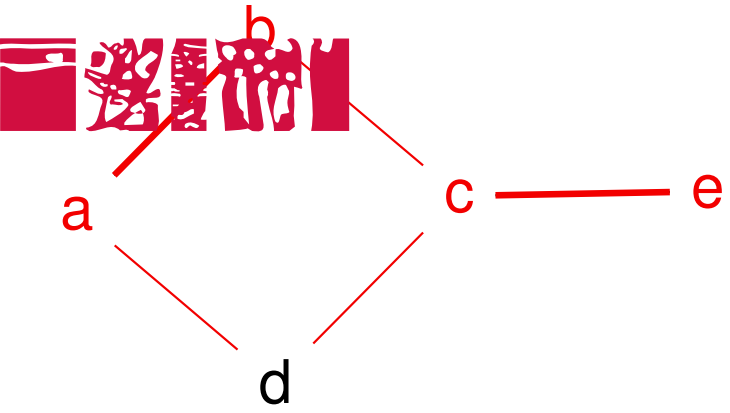

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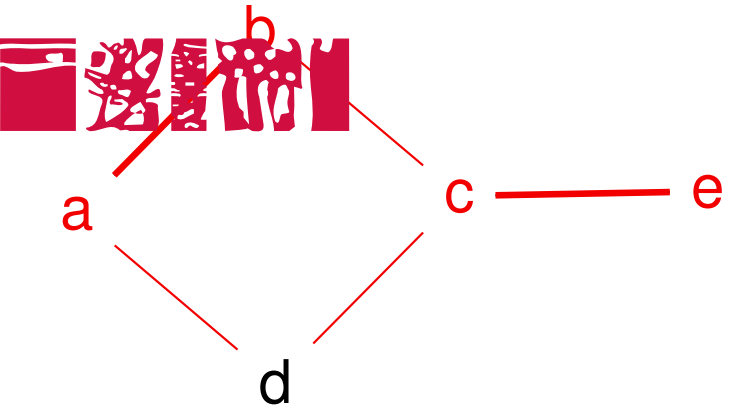

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Approximated: $\{a, b, c, e\}$, size 4

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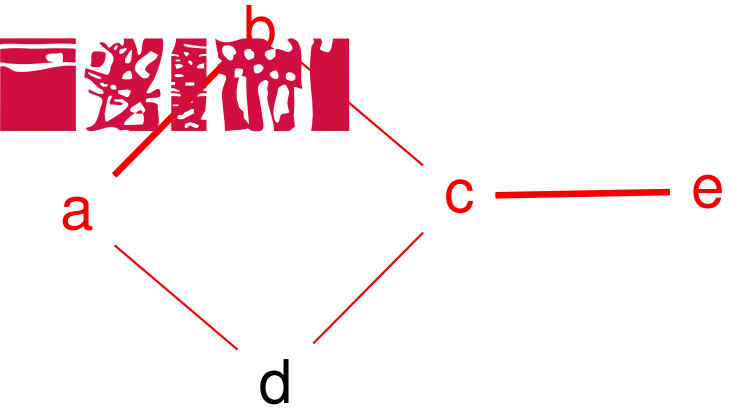

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Multiplicative 2-approximation! (twice as many nodes in the approximated cover as edges chosen, each of this edge need to be covered)

- Set Cover has a $(\ln n + O(1))$-approximation algorithm [Chv79] but is not in APX [LY94]
- Inconsistent Data Repair is in APX (but the constant depends on the dependencies) [KL09]
- Influence Maximization is in APX; it has a $(1 - 1/e - \varepsilon)$-approximation algorithm for any ε (slightly better than 63%) [KKT03]

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- Vertex Cover is not 1.3606-approximable [DS05] (!)
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- This kind of results is usually much more difficult to obtain than approximation algorithms

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- By exploiting approximation-preserving reductions between a problem and an approximable problem (in both directions); various notions of approximation-preserving, arbitrary reductions don't work

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- FPTAS (Fully Polynomial-Time Approximation Scheme): PTAS whose overall complexity depends polynomially in $1/\varepsilon$

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- Knapsack is an NP-hard problem, but there exists an FPTAS

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Often too conservative!

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- This is completely independent of m!

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Examples

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Probabilistic Query Evaluation

PRAS (Polynomial-time Randomized Approximation Scheme): there exists a polynomial-time randomized $(\varepsilon, 1/3)$-approximation for any $\varepsilon > 0$

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- FPRAS (Fully Polynomial-time Randomized Approximation Scheme): PRAS whose overall complexity depends polynomially in $1/\varepsilon$

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- See `http://webcourse.cs.technion.ac.il/236605/Spring2015/ho/WCFiles/L9%20-%20QA%20in%20PDBs.pdf` for detailed explanations

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- But one can show there is no FPRAS for arbitary SQL queries

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$\rightarrow$ Lineage: $f_2 \wedge \big((g_2 \wedge h_1) \vee (g_3 \wedge h_2)\big) = (f_2 \wedge g_2 \wedge h_1) \vee (f_2 \wedge g_3 \wedge h_2)$

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Merci.

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