CS3236: Introduction to Information Theory

Lecture 12: Introduction to Kolmogorov Complexity

April 22, 2026

Pierre Senellart

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  $110111001011101111000 1001\ldots$ (Champernowne constant):
  - (Probability theory) Yes! It has the same probability as every other possibility.
  - (Statistics) Yes! This is a normal number.

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- (Shannon's Entropy) Then your message has entropy $1,000,000$ bits and you can't compress it!
- Really? Even if my message is $1^{1000000}$ or the Champernowne constant?
- (Shannon's Entropy) Yup, nothing special in this particular message, it has the same probability as any other message under your prior.

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- First attempt: the intrinsic complexity of an object is the shortest English sentence that describes this object
- Not an effective description! English is ambiguous, and allows expressing very complex notions, how can I go from an English sentence to the corresponding object?
- Berry paradox: “the smallest positive integer not definable in fewer than twelve words”
- Why English? Seems like an arbitrary choice

- Objects can always be seen as a sequence of bits (e.g., finite sequence for integers or character strings, infinite for reals), i.e., an element of $X = \{0,1\}^{\mathbb{N}}$.
- Representations (e.g., English language sentences) can always be seen as finite sequence of bits
- Looking for a function $f : \{0,1\}^* \to X$ such that:
  - f is effective: given x, one can compute $f(x)$ (or enumerate the bits of $f(x)$ if x is finite) in a mechanical manner
  - f is generic: no dependency on an arbitrary choice
  - f is universal: all objects X "worth describing" (all finite sequences, and interesting infinite sequences) are in $f(\{0,1\}^*)$
  - f has compact representations, i.e., $C_f(y) = \min_{\substack{x \in \{0,1\}^* \\ f(x)=y}} \ell(x)$ as small as possible; $C_f(y)$ minimum-length description of y for f

Complexity of an Object

Introduction to Computability

Turing Machines

Kolmogorov Complexity

For Next Week

- Computation model: one particular function from a representation in $\{0,1\}^*$ to an element of X

Theorem

A given computation model cannot have for range all possible objects of X.

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Proof.

- Countably many finite sequences in $\{0,1\}^*$
- Uncountably many possible objects in $\{0,1\}^{\mathbb{N}}$ □

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Turing machine: transitions describing the actions of a read/write head on an infinite tape

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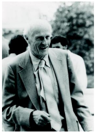
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Recursive functions: functions definable by a set of operations (constants, recursion, composition, etc.)

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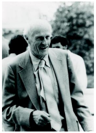

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(John von Neumann)

Von Neumann machines: code reading and writing the value of registers (variables) in an arbitrary order

Theorem

All computation models from the previous slide are equivalent, i.e., can represent the same objects.

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Thesis (Church–Turing)

These computation models exactly define the objects that a human brain, or that nature, are able to compute.

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A (deterministic) Turing machine on the alphabet $\Gamma\{0, 1, \perp\}$ where $\perp$ is a blank symbol, is defined by:

- a finite set of states Q
- an initial state $q_0 \in Q$ and a set of final states $F \subseteq Q$
- a (partial) transition function: $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \rightarrow\}$

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A transition $(q, \alpha) \mapsto (q', \alpha', \leftarrow)$ is read:

> *When the machine is in state q and reads symbol α on the read/write head, it goes into state q', writes symbol α' and moves the read/write head to the left.*

A configuration of a Turing machine is:

- the finite tape content (with infinitely many ⊥ left and right)
- the position of the head on the tape
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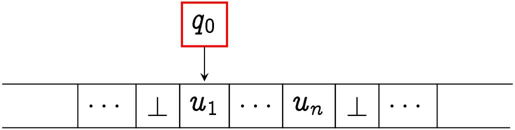

Initial configuration for $u = u_1 \ldots u_n$

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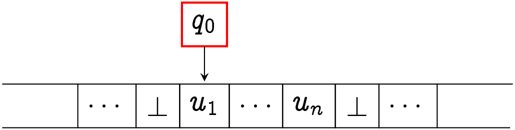

Initial configuration for $u = u_1 \dots u_n$

The result $T(u)$ of the computation of the Turing machine T on u is the content of the tape (say, disregarding blanks) once the Turing machine reaches a final state $q_f \in F$ starting from the initial configuration for u and following the transitions.

Let $n \in \mathbb{N}$, binary representation $n_{l-1}n_{l-2}\dots n_0$ ($n = \sum_{k=0}^{l-1} n_k 2^k$).

$Q = \{q^{\mathrm{in}}_{l-1} \dots q^{\mathrm{in}}_0, q_{\mathrm{s}}, q^0_{\rightarrow}, q^1_{\rightarrow}, q_{\leftarrow}, q_{\mathrm{dec}}, q_{\mathrm{w}}, q_{\mathrm{f}}\}$, $q_0 = q^{\mathrm{in}}_{l-1}$, $F = \{q_{\mathrm{f}}\}$

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δ :

	0	1	$\perp$	
q^{in}_k			$q^{\mathrm{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\mathrm{s}}, n_0, \rightarrow$	
q_{s}	$q_{\mathrm{s}}, 0, \rightarrow$	$q_{\mathrm{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
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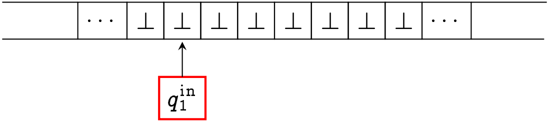

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q^{in}_0

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q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q_{\rightarrow}^0, \perp, \rightarrow$	
$q_{\rightarrow}^0$	$q_{\rightarrow}^1, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q_{\rightarrow}^1$		$q_{\rightarrow}^0, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
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$q^0_{\rightarrow}$

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⋯	$\bot$	1	1	$\bot$	0	$\bot$	$\bot$	$\bot$	⋯

q_{dec}

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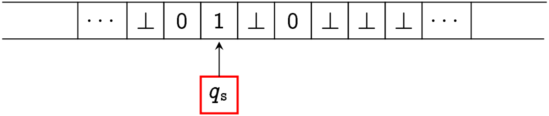

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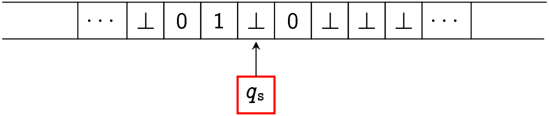
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	



$n = 2$

Let $n \in \mathbb{N}$, binary representation $n_{l-1}n_{l-2}\dots n_0$ $(n = \sum_{k=0}^{l-1} n_k 2^k)$.

$Q = \{q^{\text{in}}_{l-1} \dots q^{\text{in}}_0, q_{\text{s}}, q^0_{\rightarrow}, q^1_{\rightarrow}, q_{\leftarrow}, q_{\text{dec}}, q_{\text{w}}, q_{\text{f}}\}$, $q_0 = q^{\text{in}}_{l-1}$, $F = \{q_{\text{f}}\}$

δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	

$\cdots$	$\perp$	0	1	$\perp$	0	$\perp$	$\perp$	$\perp$	$\cdots$
					↑ $q^0_{\rightarrow}$				

$n = 2$

Let $n \in \mathbb{N}$, binary representation $n_{l-1}n_{l-2}\dots n_0$ ($n = \sum_{k=0}^{l-1} n_k 2^k$).

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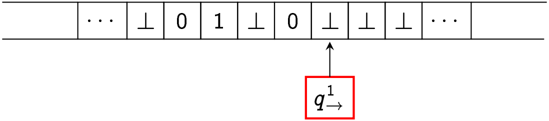
δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	



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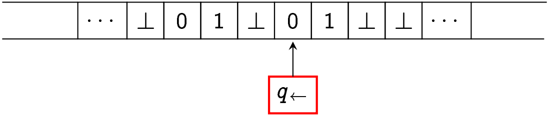
δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	



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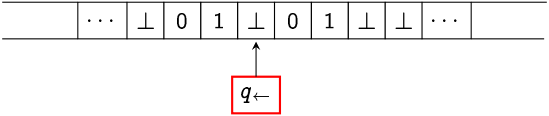
δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	



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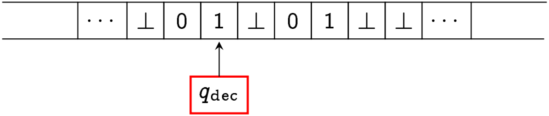
δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	



$n = 2$

Let $n \in \mathbb{N}$, binary representation $n_{l-1}n_{l-2}\ldots n_0$ ($n = \sum_{k=0}^{l-1} n_k 2^k$).

$Q = \{q^{\text{in}}_{l-1} \ldots q^{\text{in}}_0, q_{\text{s}}, q^0_{\rightarrow}, q^1_{\rightarrow}, q_{\leftarrow}, q_{\text{dec}}, q_{\text{w}}, q_{\text{f}}\}$, $q_0 = q^{\text{in}}_{l-1}$, $F = \{q_{\text{f}}\}$

δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	

$\cdots$	$\perp$	0	0	$\perp$	0	1	$\perp$	$\perp$	$\cdots$

q_{s}

$n = 2$

Let $n \in \mathbb{N}$, binary representation $n_{l-1}n_{l-2}\dots n_0$ ($n = \sum_{k=0}^{l-1} n_k 2^k$).

$Q = \{q^{\text{in}}_{l-1} \dots q^{\text{in}}_0, q_{\text{s}}, q^0_{\rightarrow}, q^1_{\rightarrow}, q_{\leftarrow}, q_{\text{dec}}, q_{\text{w}}, q_{\text{f}}\}$, $q_0 = q^{\text{in}}_{l-1}$, $F = \{q_{\text{f}}\}$

δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	

…	$\perp$	0	0	$\perp$	0	1	$\perp$	$\perp$	…

↑

$q^0_{\rightarrow}$

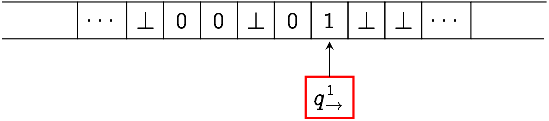
$n = 2$

Let $n \in \mathbb{N}$, binary representation $n_{l-1}n_{l-2}\dots n_0$ ($n = \sum_{k=0}^{l-1} n_k 2^k$).

[…] $\dots q_0^{\text{in}}, q_s, q_{\rightarrow}^0, q_{\rightarrow}^1, q_{\leftarrow}, q_{\text{dec}}, q_w, q_f\}$, $q_0 = q_{l-1}^{\text{in}}$, $F = \{q_f\}$

δ :

	0	1	$\perp$	
q_k^{in}			$q_{k-1}^{\text{in}}, n_k, \rightarrow$	$k > 0$
q_0^{in}			$q_s, n_0, \rightarrow$	
q_s	$q_s, 0, \rightarrow$	$q_s, 1, \rightarrow$	$q_{\rightarrow}^0, \perp, \rightarrow$	
$q_{\rightarrow}^0$	$q_{\rightarrow}^1, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q_{\rightarrow}^1$		$q_{\rightarrow}^0, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_s, 0, \rightarrow$	$q_w, \perp, \rightarrow$	
q_w	$q_w, \perp, \rightarrow$		$q_f, \perp, \rightarrow$	



$n = 2$

Let $n \in \mathbb{N}$, binary representation $n_{l-1}n_{l-2}\dots n_0$ ($n = \sum_{k=0}^{l-1} n_k 2^k$).

$Q = \{q^{\text{in}}_{l-1} \dots q^{\text{in}}_0, q_{\text{s}}, q^0_{\rightarrow}, q^1_{\rightarrow}, q_{\leftarrow}, q_{\text{dec}}, q_{\text{w}}, q_{\text{f}}\}$, $q_0 = q^{\text{in}}_{l-1}$, $F = \{q_{\text{f}}\}$

δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	

$\cdots$	$\perp$	0	0	$\perp$	0	1	$\perp$	$\perp$	$\cdots$

$\uparrow$

$q^0_{\rightarrow}$

$n = 2$

Let $n \in \mathbb{N}$, binary representation $n_{l-1}n_{l-2}\dots n_0$ ($n = \sum_{k=0}^{l-1} n_k 2^k$).

$Q = \{q^{\text{in}}_{l-1} \dots q^{\text{in}}_0, q_{\text{s}}, q^0_{\rightarrow}, q^1_{\rightarrow}, q_{\leftarrow}, q_{\text{dec}}, q_{\text{w}}, q_{\text{f}}\}$, $q_0 = q^{\text{in}}_{l-1}$, $F = \{q_{\text{f}}\}$

δ :

	0	1	⊥	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \bot, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \bot, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \bot, \rightarrow$	
q_{w}	$q_{\text{w}}, \bot, \rightarrow$		$q_{\text{f}}, \bot, \rightarrow$	

···	⊥	0	0	⊥	0	1	0	⊥	···

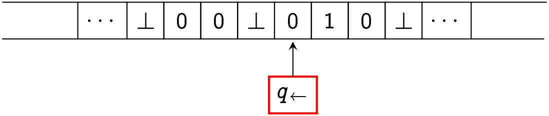
$q_{\leftarrow}$

$n = 2$

Let $n \in \mathbb{N}$, binary representation $n_{l-1}n_{l-2}\ldots n_0$ ($n = \sum_{k=0}^{l-1} n_k 2^k$).
$Q = \{q^{\text{in}}_{l-1}\ldots q^{\text{in}}_0, q_{\text{s}}, q^0_{\rightarrow}, q^1_{\rightarrow}, q_{\leftarrow}, q_{\text{dec}}, q_{\text{w}}, q_{\text{f}}\}$, $q_0 = q^{\text{in}}_{l-1}$, $F = \{q_{\text{f}}\}$

δ :

	0	1	$\bot$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \bot, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \bot, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \bot, \rightarrow$	
q_{w}	$q_{\text{w}}, \bot, \rightarrow$		$q_{\text{f}}, \bot, \rightarrow$	



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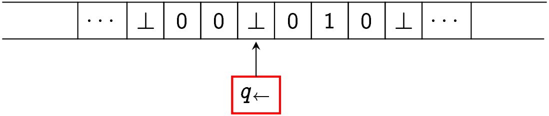
δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	



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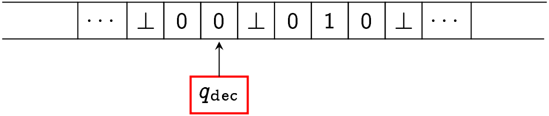
δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_\text{s}, n_0, \rightarrow$	
q_s	$q_\text{s}, 0, \rightarrow$	$q_\text{s}, 1, \rightarrow$	$q^0_\rightarrow, \perp, \rightarrow$	
$q^0_\rightarrow$	$q^1_\rightarrow, 0, \rightarrow$		$q_\leftarrow, 0, \leftarrow$	
$q^1_\rightarrow$		$q^0_\rightarrow, 1, \rightarrow$	$q_\leftarrow, 1, \leftarrow$	
$q_\leftarrow$	$q_\leftarrow, 0, \leftarrow$	$q_\leftarrow, 1, \leftarrow$	$q_\text{dec}, \perp, \leftarrow$	
q_dec	$q_\text{dec}, 1, \leftarrow$	$q_\text{s}, 0, \rightarrow$	$q_\text{w}, \perp, \rightarrow$	
q_w	$q_\text{w}, \perp, \rightarrow$		$q_\text{f}, \perp, \rightarrow$	



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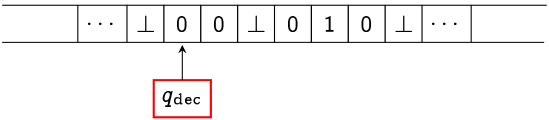
δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	



$n = 2$

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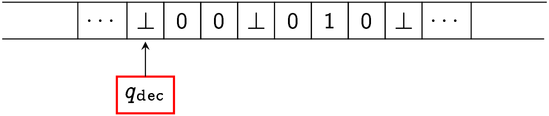
δ :

	0	1	$\perp$	
q^{in}_k			$q^{\text{in}}_{k-1}, n_k, \rightarrow$	$k > 0$
q^{in}_0			$q_{\text{s}}, n_0, \rightarrow$	
q_{s}	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{s}}, 1, \rightarrow$	$q^0_{\rightarrow}, \perp, \rightarrow$	
$q^0_{\rightarrow}$	$q^1_{\rightarrow}, 0, \rightarrow$		$q_{\leftarrow}, 0, \leftarrow$	
$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
$q_{\leftarrow}$	$q_{\leftarrow}, 0, \leftarrow$	$q_{\leftarrow}, 1, \leftarrow$	$q_{\text{dec}}, \perp, \leftarrow$	
q_{dec}	$q_{\text{dec}}, 1, \leftarrow$	$q_{\text{s}}, 0, \rightarrow$	$q_{\text{w}}, \perp, \rightarrow$	
q_{w}	$q_{\text{w}}, \perp, \rightarrow$		$q_{\text{f}}, \perp, \rightarrow$	



$n = 2$

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$q^1_{\rightarrow}$		$q^0_{\rightarrow}, 1, \rightarrow$	$q_{\leftarrow}, 1, \leftarrow$	
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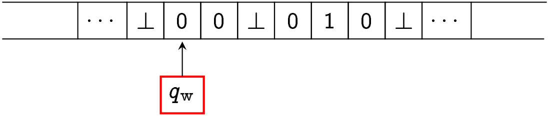

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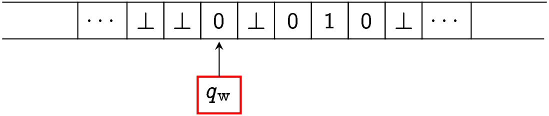

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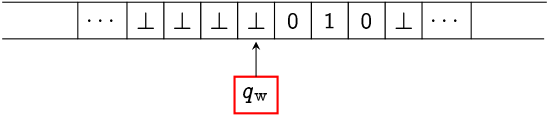

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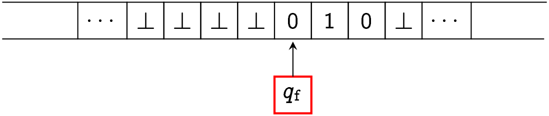

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- Often convenient to have a prefix-free encoding of arbitrary (positive) integers, to allow their use in a code of a more complex object
- For a positive integer k, let $\tilde{k}$ be the binary representation of k without initial 1.
- One defines the following family of codes for positive integers:

$$\begin{cases} E_0(k) := 0^k 1 \\ E_i(k) := E_{i-1}(\lfloor \log k \rfloor)\tilde{k} & i > 0 \end{cases}$$

- For example, for $k = 9$:

i	$E_i(k)$
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- $\ell(E_0(k)) = k + 1$,
  $\ell(E_1(k)) = 2\lfloor \log k \rfloor + 1$,
  $\ell(E_2(k)) = 2\lfloor \log \lfloor \log k \rfloor \rfloor + \lfloor \log k \rfloor + 1$

- Turing machines so far only compute finite sequences
- A Turing machine T computes an infinite sequence y on input x if and only if for all $k \in \mathbb{N}^*$, $T(E_2(k)x)$ is the k-bit prefix of y.

- $T = (Q, q_0, F, \delta)$
- Let $s = \lceil \log |Q| \rceil$, r the size of the domain of δ, $f = |F|$.
- We encode each state of Q as $e(Q)$, an s-bit string.
- We encode each tabe symbol γ of Γ as $e(\gamma)$, a 2-bit string.
- We encode $\alpha \in \{\leftarrow, \rightarrow\}$ as $e(\alpha)$, a 1-bit string.
- We enumerate rules of $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \rightarrow\}$ as $(q_1, \gamma_1, q'_1, \gamma'_1, \alpha_1) \dots (q_r, \gamma_r, q'_r, \gamma'_r, \alpha_r)$.
- We encode T as:

$$E(T) := E_1(s)E_1(r)E_1(f)e(q_1)e(\gamma_1)e(q'_1)e(\gamma'_1)e(\alpha_1)\dots$$
$$e(q_r)e(\gamma_r)e(q'_r)e(\gamma'_r)e(\alpha_r)e(q_{\mathrm{f}_1})\dots e(q_{\mathrm{f}_f}) = C$$

with conventionally $E(q_0) = 0^s$ and $F = \{q_{\mathrm{f}_1}, \dots, q_{\mathrm{f}_f}\}$

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- $\ell(E(T)) \leq r(2s + 5) + 2\log(rsf) + fs + 3$

n=2 $|Q| = 9$, $s = 4$, $f = 1$, $r = 17$

What does the encoding of T look like?

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What does the encoding of T look like?

General case $|Q| = \lfloor \log(n) \rfloor + 7$, $s = O(\log \log n)$, $r = \lfloor \log(n) \rfloor + 16$, $f = 1$

$$\ell(E(T)) = O(\log n \times \log \log n)$$

Proposition

There exists a Turing machine T_{Check} *such that* $T_{\text{Check}}(x) = 1$ *if* x *is of the form* $E(T)$ *for any Turing machine* T*, and* 0 *otherwise.*

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Proposition

There exists a Turing machine U such that, for all Turing machines T and finite sequences x, $U(E(T)x) = T(x)$.

U is called a universal Turing machine.

Proposition (Turing)

There is no Turing machine H such that $H(w) = 1$ if $w = xy$ with x of the form $E(T)$ for some Turing machine T and $T(y)$ finishes, and $H(w) = 0$ otherwise.

Complexity of an Object

Introduction to Computability

Turing Machines

Kolmogorov Complexity

For Next Week

Definition

Given a Turing machine T, the Kolmogorov complexity of an object $y \in X$ relative to T is $C_T(y) = \min_{\substack{x \in \{0,1\}^* \\ T(x)=y}} \ell(x)$.

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Theorem (Invariance)

Let U_1 and U_2 be two universal Turing machines. Then $|C_{U_1}(y) - C_{U_2}(y)| \leq c_{U_1,U_2}$ where c_{U_1,U_2} is a constant independent of y.

We choose one such reference Turing machine and simply write $C(x)$. The invariance theorem guarantees that such a choice is generic.

- Objects can always be seen as a sequence of bits (e.g., finite sequence for integers or character strings, infinite for reals), i.e., an element of $X = \{0,1\}^{\mathbb{N}}$.
- Representations (e.g., English language sentences) can always be seen as finite sequence of bits
- Looking for a function $f : \{0,1\}^* \to X$ such that:
  - f is effective: given x, one can compute $f(x)$ (or enumerate the bits of $f(x)$ if x is finite) in a mechanical manner
  - f is generic: no dependency on an arbitrary choice
  - f is universal: all objects X "worth describing" (all finite sequences, and interesting infinite sequences) are in $f(\{0,1\}^*)$
  - f has compact representations, i.e., $C_f(y) = \min_{\substack{x\in\{0,1\}^* \\ f(x)=y}} \ell(x)$ as small as possible; $C_f(y)$ minimum-length description of y for f

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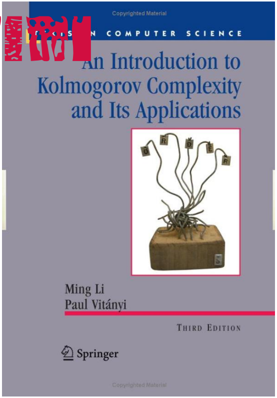

Textbook on Kolmogorov complexity, chapter 1 (not compulsory, but useful to go into more detail).

Homework due by Monday 2pm next week (last homework!), published later tonight.