CS3236: Homework 7

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Due Monday, October 13, 2pm

Submission: Submission can be made in person before the beginning of the lecture, or by upload to the IVLE Workbin. You may submit handwritten answers (possibly scanned), but they must be **clearly readable**. Answers that are not readable may receive only partial points. Late answers are still accepted on Tuesday by 2pm (on the IVLE Workbin only), but receive only 50% of the possible points. Any later submissions receive no points. Assignments are marked out of 20 points.

1. (4 points) Consider a channel given by the transition probabilities $\Pr(y \mid x)$ with capacity C. Suppose that a friendly statistician offers to process the output for you by applying a function g, giving you $\tilde{y} = g(y)$.
   a) (2 points) Show that your friend's strategy can only *decrease* the capacity of the channel.
   b) (2 points) Under which conditions (on g) is the capacity indeed strictly decreased?
2. (6 points) Do exercise 10.12 in the textbook.
3. (6 points) Consider a binary symmetric channel with output $y_j = x_j + z_j \mod 2$ where z_j is chosen with probability $f = \Pr(z_j = 1)$. Let us now assume that when we use the channel n times, the channel has memory. That is, if the channel chooses $z_1, \ldots, z_n$ then the marginal probabilities $\Pr(z_j = 1) = f$, but that the $z_1, \ldots, z_n$ are not independent of each other (but still independent of the inputs $x^n = x_1, \ldots, x_n$). That is, $\Pr(z_1, \ldots, z_n)$ is *not* a product distribution. We know that the capacity of the memoryless binary symmetric channel is given by $C = 1 - H_2(f)$. Show that the capacity of the channel with memory obeys $C_{\text{mem}} = \max_{\Pr(x_1,\ldots,x_n)} I(X_1, \ldots, X_n; Y_1, \ldots, Y_n) \geq nC$. How do you intepret your result?
4. (4 points) Consider some variants of the noisy typewriter.
   a) (1 point) If pushing a key results in precisely outputting the associated letter, what is the capacity C in bits?
   b) Now suppose that pushing a key results in outputting that letter or the next with equal probability. That is, $A \to (A, B)$, $B \to (B, C)$ and so on.
      i. (2 points) What is the capacity C in bits?
      ii. (1 point) What is the highest rate code with block length one that you can find that achieves *zero* probability of error for the same channel?