CS3236: Homework 6

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Due Monday, September 29, 2pm

Submission: Submission can be made in person before the beginning of the lecture, or by upload to the IVLE Workbin. You may submit handwritten answers (possibly scanned), but they must be **clearly readable**. Answers that are not readable may receive only partial marks. Late answers are still accepted on Tuesday by 2pm (on the IVLE Workbin only), but receive only 50% of the possible marks. Any later submissions receive no marks. Assignments are marked out of 20 points.

1. (6 points) Consider the following probability distribution $p(x, y)$ given by $p(1, 1) = 0$, $p(2, 1) = 3/4$, $p(1, 2) = 1/8$ and $p(2, 2) = 1/8$.
   a) (2 points) Compute the entropies $H(X)$, $H(Y)$ as well as $H(X \mid Y)$ and $H(Y \mid X)$. Verify that conditioning reduces entropy.
   b) (2 points) Compute the entropies $H(X \mid Y = y)$ for all $y \in \{1, 2\}$. Discuss whether conditioning reduces entropy for all $y \in \{1, 2\}$.
   c) (2 points) Compute the mutual informations $I(X; Y)$ and $I(Y; X)$ independently and verify the symmetry of mutual information.

2. (5 points) Let $\mathcal{N}_1 = BSC(f_1)$ and $\mathcal{N}_2 = BSC(f_2)$ be two binary symmetric channels with bit flip probabilities f_1 and f_2 respectively. Imagine a new channel $\mathcal{N} = \mathcal{N}_2 \circ \mathcal{N}_1$ formed by composing the two individual channels one after the other. That is, we input x into the first channel, the output y_1 of the first channel is immediately input to the second channel, and finally we receive the output of the second channel.
   a) (1 points) Draw the transition diagram for the combined channel.
   b) (3 points) What is the capacity of the combined channel for general f_1 and f_2?
   c) (1 point) Evaluate the capacity for $f_1 = 1/3$ and $f_2 = 1/10$.

3. (9 points) Imagine a channel that takes as inputs symbols $\mathcal{X} = \{0, 1, \ldots, 6\}$, and outputs symbols $\mathcal{Y} = \{0, 1, \ldots, 6\}$ according to the rule $Y = X + Z \bmod 7$ where the channel adds a randomly chosen Z with probabilities $P_Z(1) = P_Z(2) = P_Z(3) = 1/3$.
   a) (1 point) Write down the transition diagram and the probabilities for this channel.
   b) (2 points) Compute $H(Z)$ and $H(Z \mid X)$.
   c) (2 points) Show that $H(Y \mid X) = H(Z \mid X)$.

d) (2 points) Compute the capacity of our channel. Show that the capacity is attained when the channel's input X is chosen uniformly at random, i.e., $P_X(x) = 1/7$.

e) (2 points) Find a code that transmits 1 bit per channel use without error. Compare the rate of this code to the capacity.

4. (Bonus) Suppose we hold as information a random variable Y, and want to guess the value of a (correlated) random variable X. Intuitively, $H(X \mid Y)$ tells us something about the uncertainty of X given Y, but how does this relate to a probability of guessing? Imagine that from Y we try to derive an estimate $\tilde{X}$ of X. That is, we let $\tilde{X} = g(Y)$ for some function g of Y. Let us define the probability of error as $P_e = \Pr[X \neq \tilde{X}]$. Let us also use $|X|$ to denote the number of possible values that X can take.

   Fano's inequality states that $H_2(P_e) + P_e \log(|X| - 1) \geq H(X \mid Y)$ for any function g that we might apply. Prove this inequality. (Hint: define a new random variable E for error that takes on value 1 if $X \neq \tilde{X}$ and 0 otherwise. Consider $H(E, X \mid Y)$.)