Graph

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Agenda

- **Graph Representation**
- **DFS**
- **BFS**
- **Dijkstra**
- **A* Search**
- **Bellman-Ford**
- **Floyd-Warshall**
- **Iterative? Non-iterative?**
- **MST**
- **Flow – Edmond-Karp**

Graph Representation

- **Adjacency Matrix**
  - **`bool way[100][100];`**
  - **`cin >> i >> j;`**
  - **`way[i][j] = true;`**

Graph Representation

- **Adjacency Linked List**
  - **`vector<int> v[100];`**
  - **`cin >> i >> j;`**
  - **`v[i].push_back(j);`**

Graph Representation

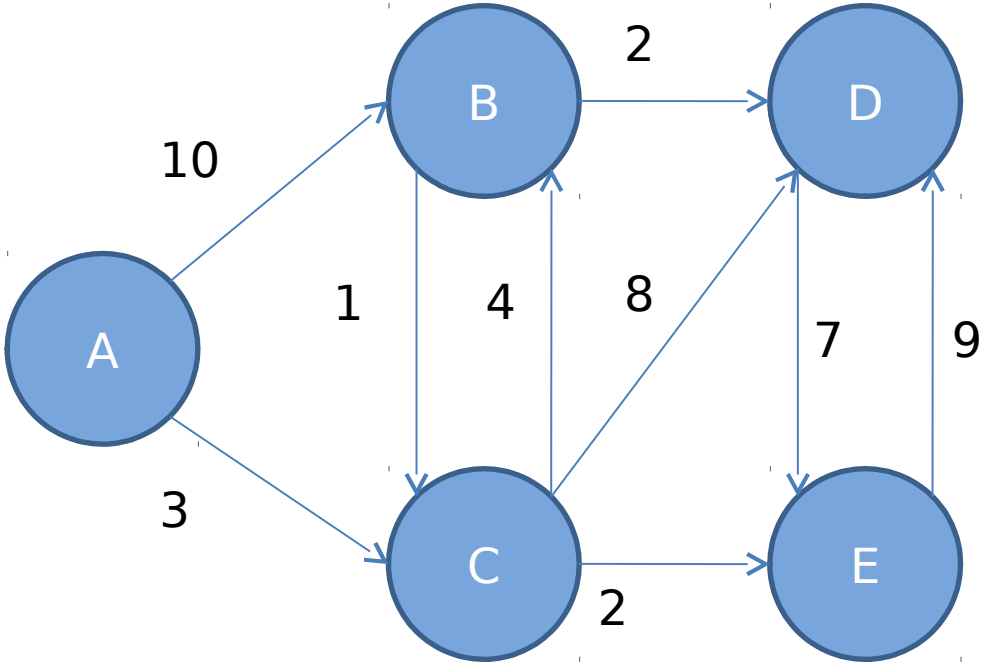
- **Edge List**

```
struct Edge {
  int Start_Vertex, End_Vertex;
} edge[10000];
cin >> edge[i].Start_Vertex
  >> edge[i].End_Vertex;
```

Graph Representation

	Adjacency Matrix	Adjacency Linked List	Edge List
Memory Storage	**$O(V^2)$**	**O(V+E)**	**O(V+E)**
Check whether (u,v) is an edge	**O(1)**	**O(deg(u))** O(log deg(u)) if sorted	**O(E)** O(log E log deg(u)) if sorted
Find all adjacent vertices of a vertex u	**O(V)**	**O(deg(u))**	**O(E)** O(log E deg(u)) if sorted
deg(u): the number of edges connecting vertex *u*			

Graph Theory



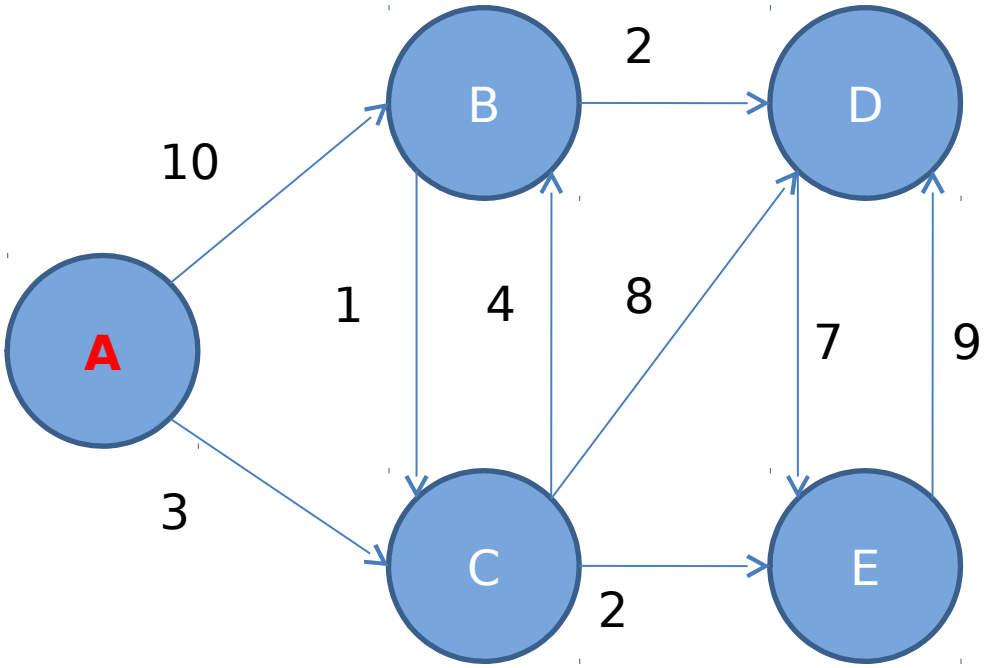
Graph Theory

- **Mission: To go from Point A to Point E**
- **How?**

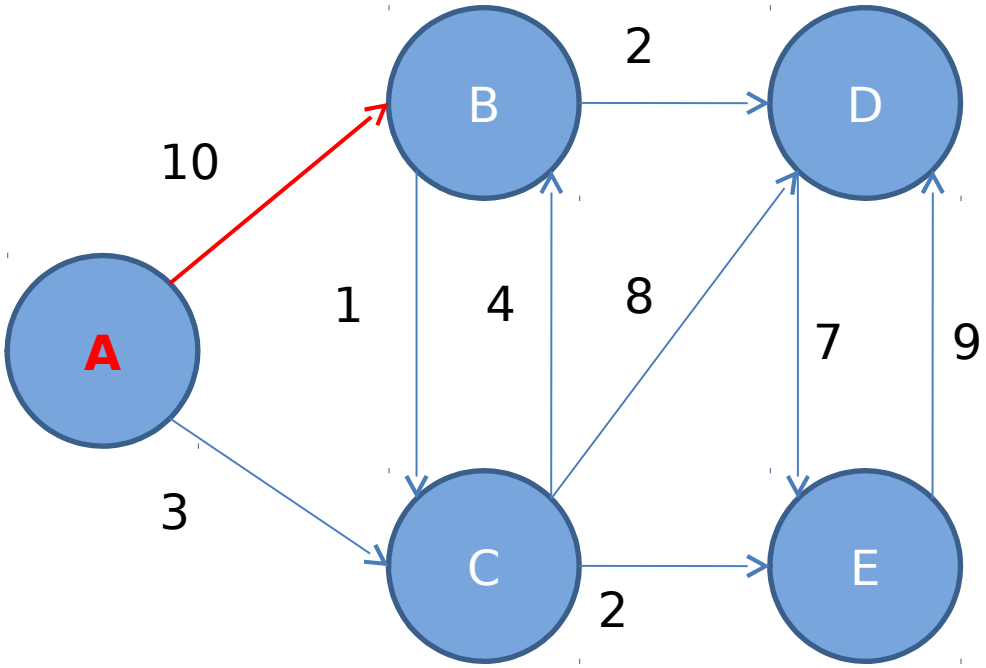
Depth First Search (DFS)

- **Structure to use: Stack**

Depth First Search (DFS)



Depth First Search (DFS)



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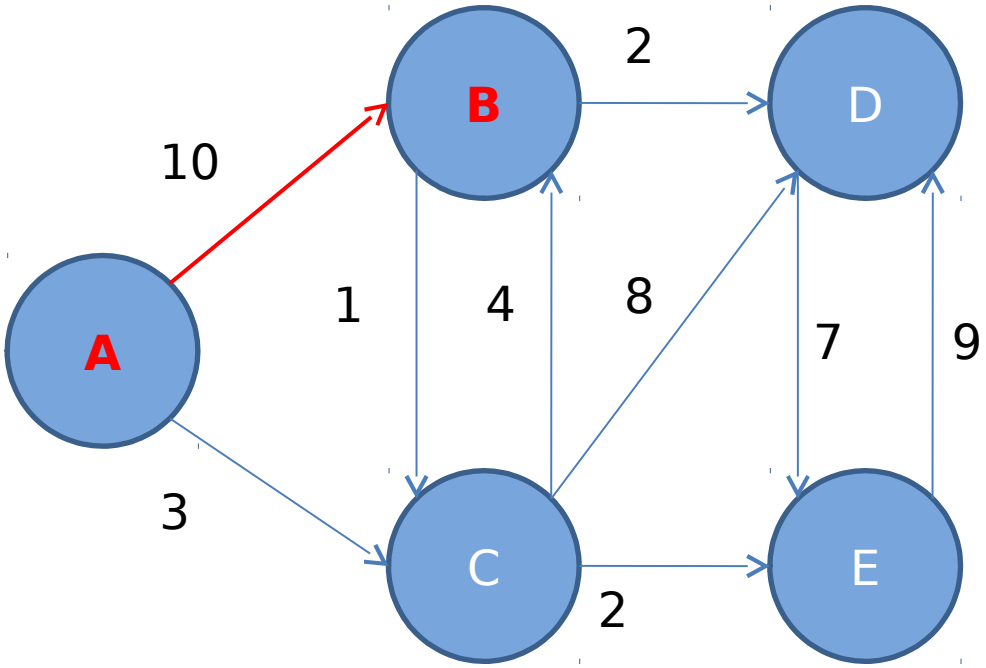
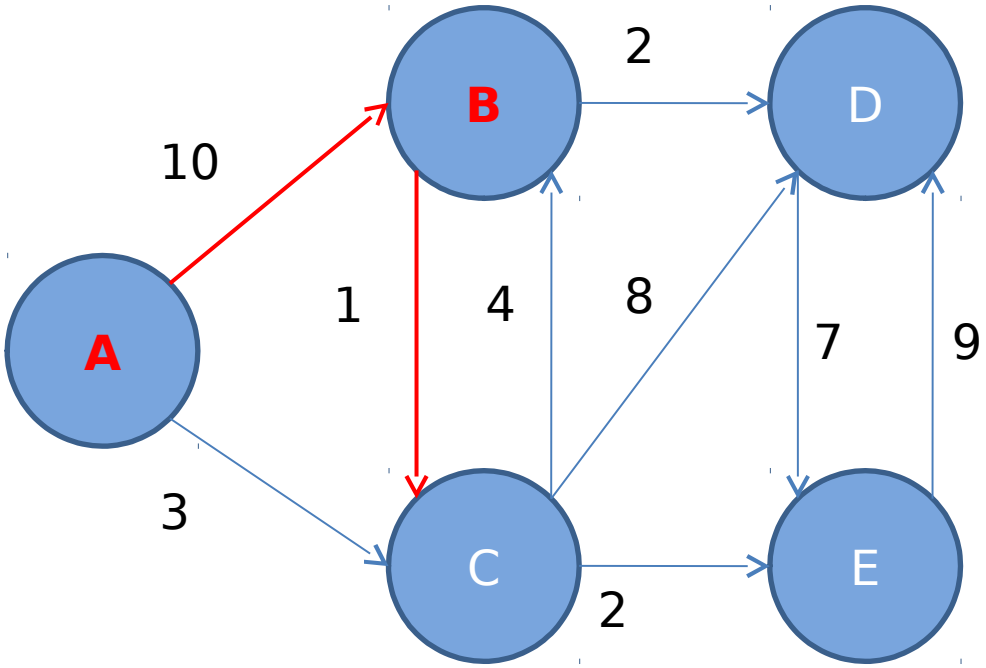


Depth First Search (DFS)



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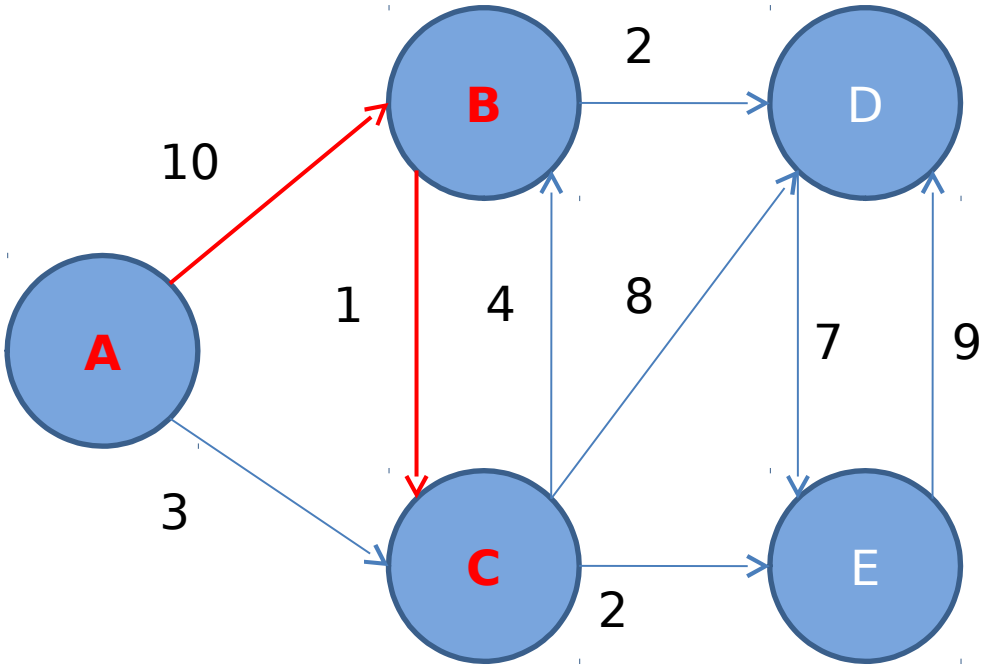
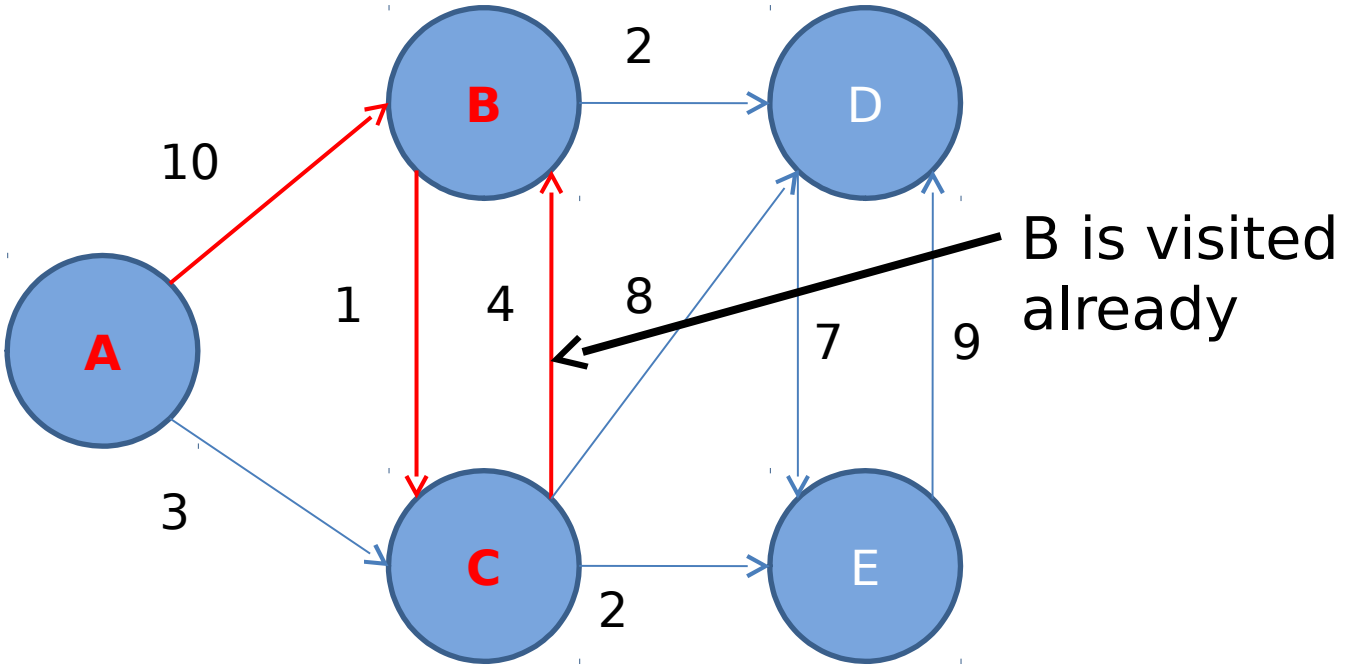
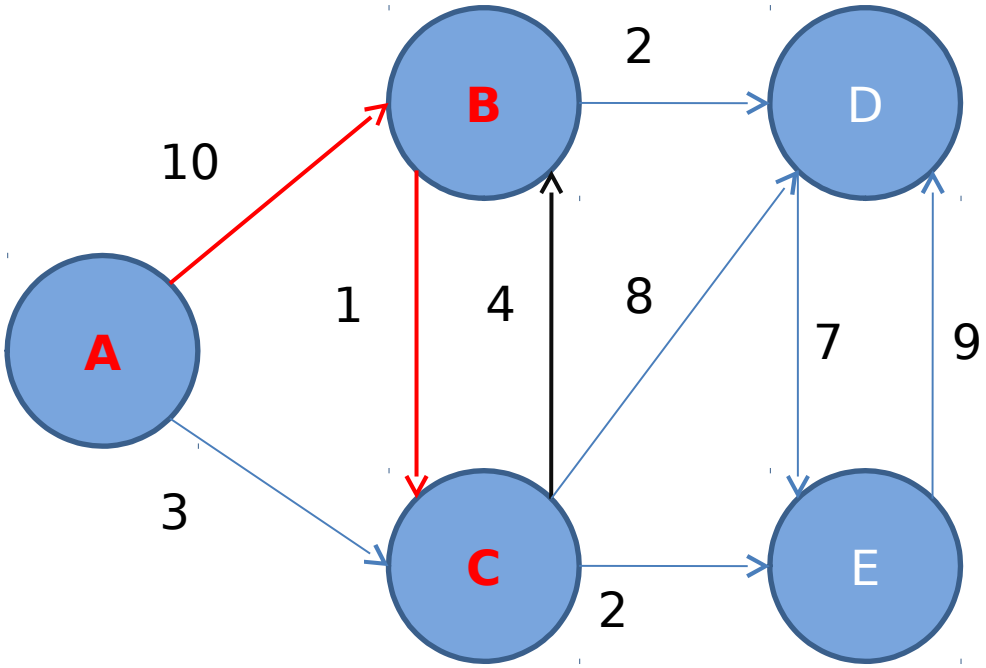


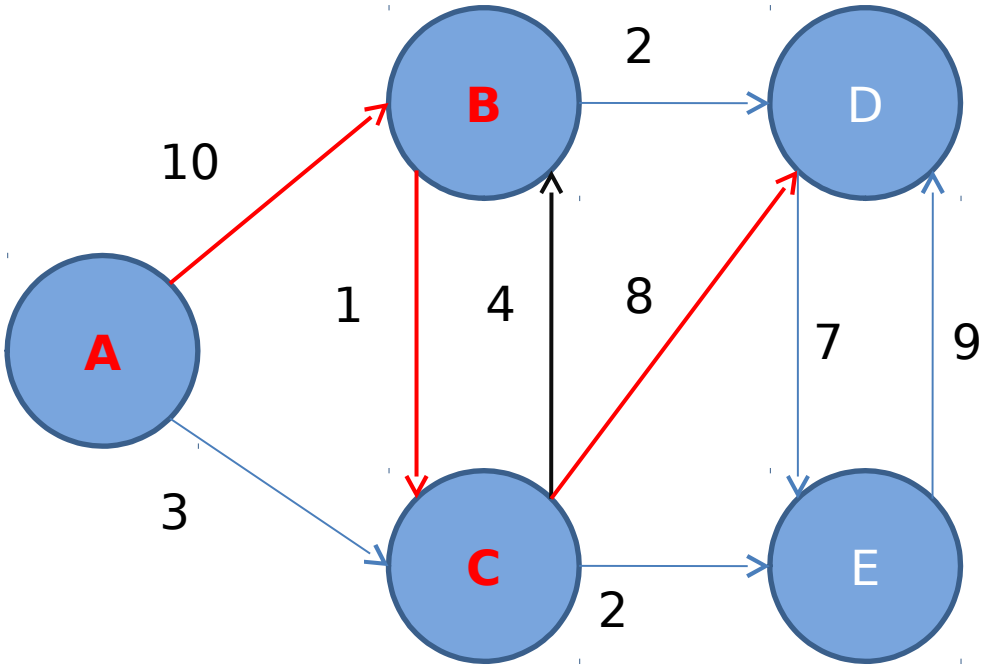
Depth First Search (DFS)



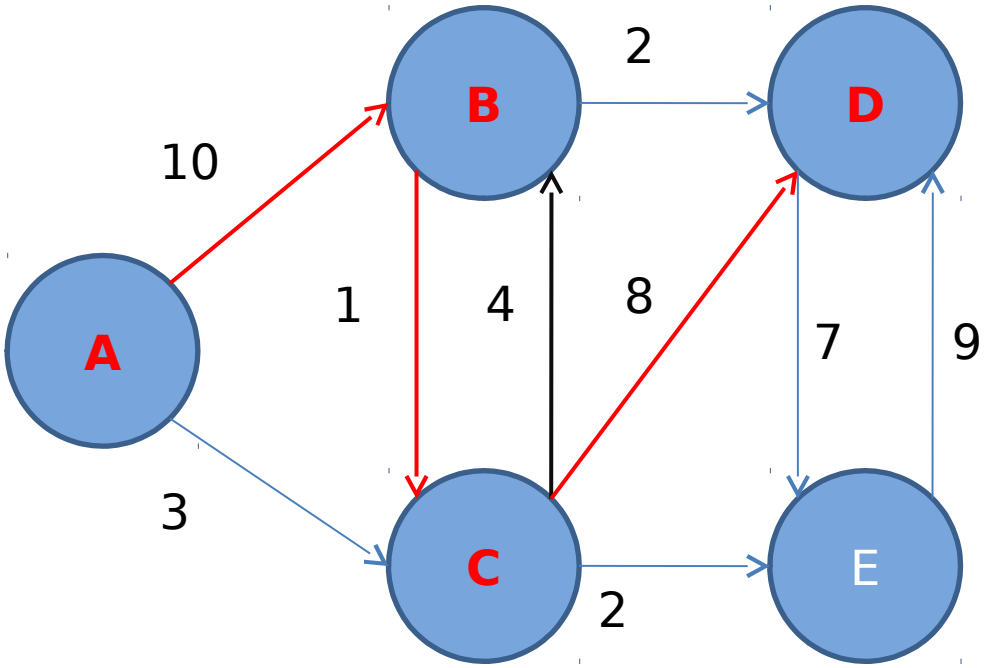
Depth First Search (DFS)



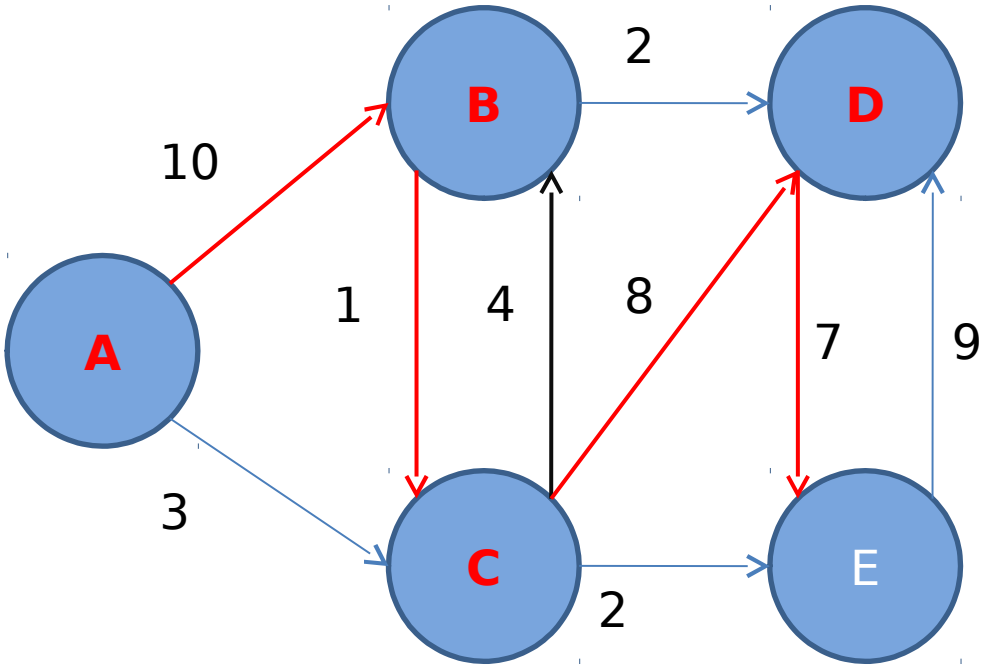
Depth First Search (DFS)



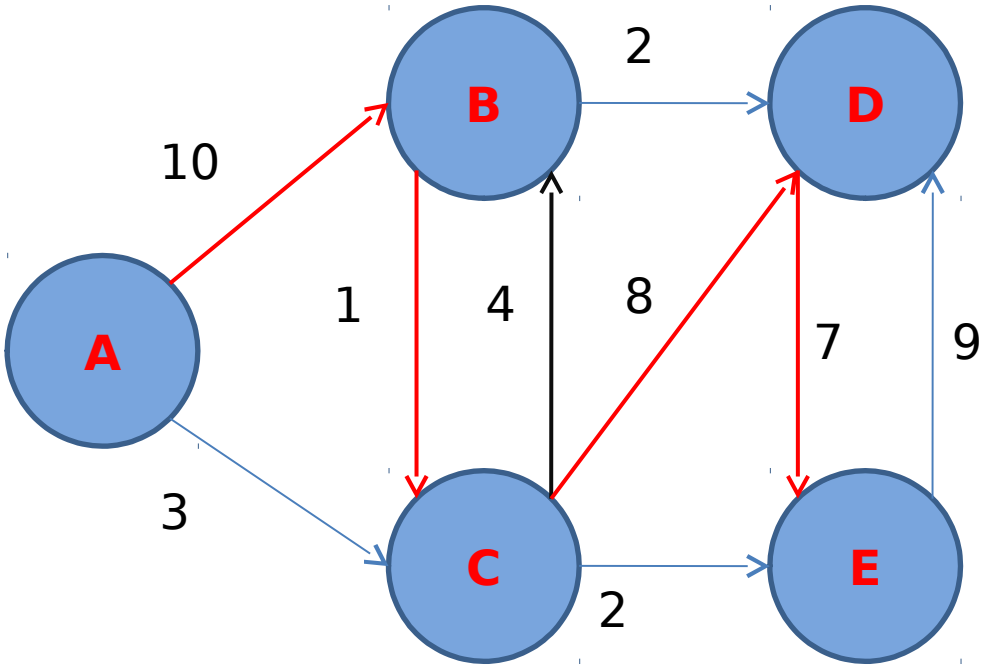
Depth First Search (DFS)



Depth First Search (DFS)



Depth First Search (DFS)



We find a path from Point A to Point E, but …

Depth First Search (DFS)

```
stack<int> s;
bool Visited[MAX];

void DFS(int u) {
    s.push(u);
    if (u == GOAL) {
        for (int x = 0; x < s.size(); x++)
            printf("%d ", s[x]);
        return ;
    }
    for (int x = 0; x < MAX; x++)
        if (!Visited[x]) {
            Visited[x] = true;
            DFS(x);
            Visited[x] = false;
        }
    s.pop();
}
```

```
int main() {
    memset(Visited, 0,
                sizeof(Visited));
    // Input Handling (GOAL = ?)
    DFS(0);
}
```

Very Important
It needs to be restored for other points to traverse, or the path may not be found

Depth First Search (DFS)

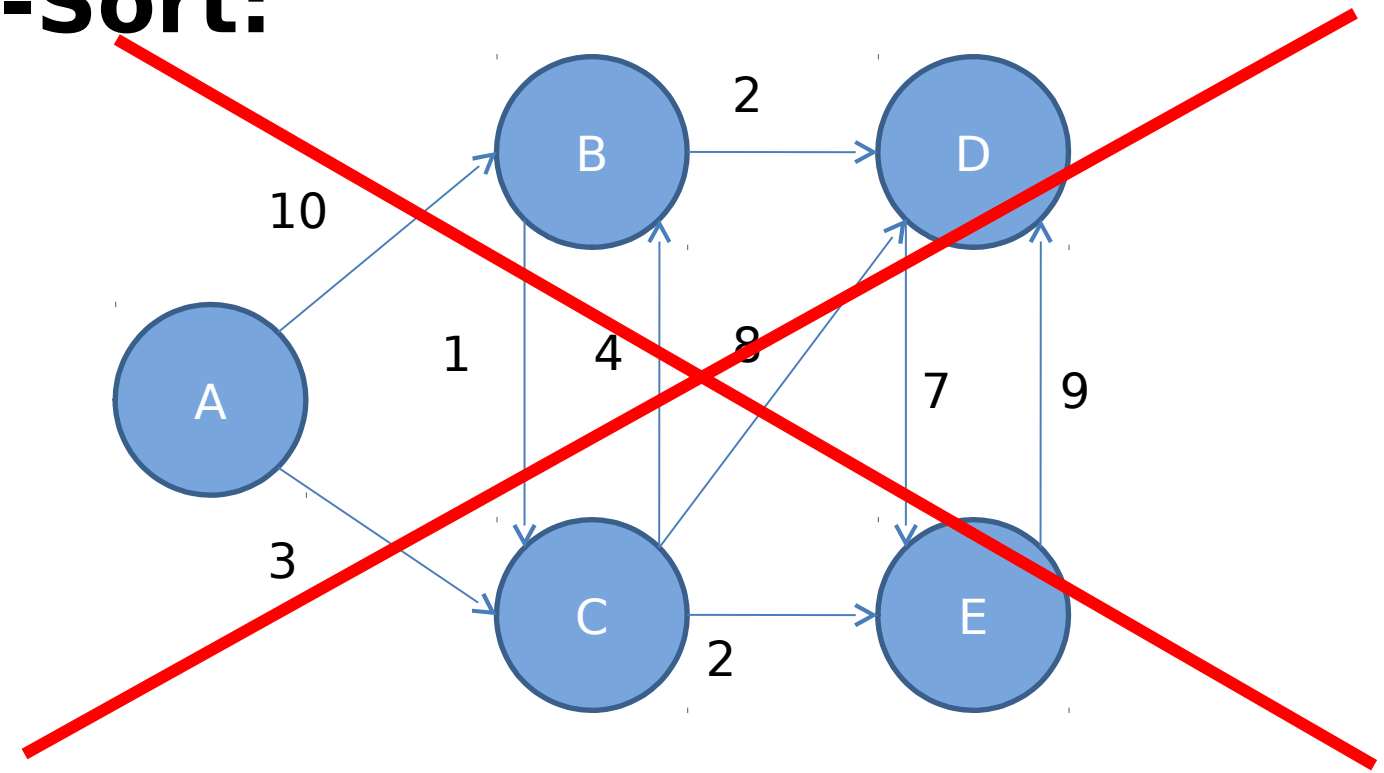
- **Usage:**
  - **Finding a path from start to destination (NOT RECOMMENDED – TLE)**
  - **Topological Sort (T-Sort)**
  - **Strongly-Connected Components (SCC)**
  - **Detect cycles**

Depth First Search (DFS)

- **Topological Sort**
  - **Directed Acyclic Graph (DAG)**
  - **Find the order of nodes such that for each node, every parent is before that node on the list**
  - **Dumb method: Check for root of residual graph (Do it n times)**
  - **Better Method: Reverse of finishing time**
  - **Start from all vertices!**

Depth First Search (DFS)

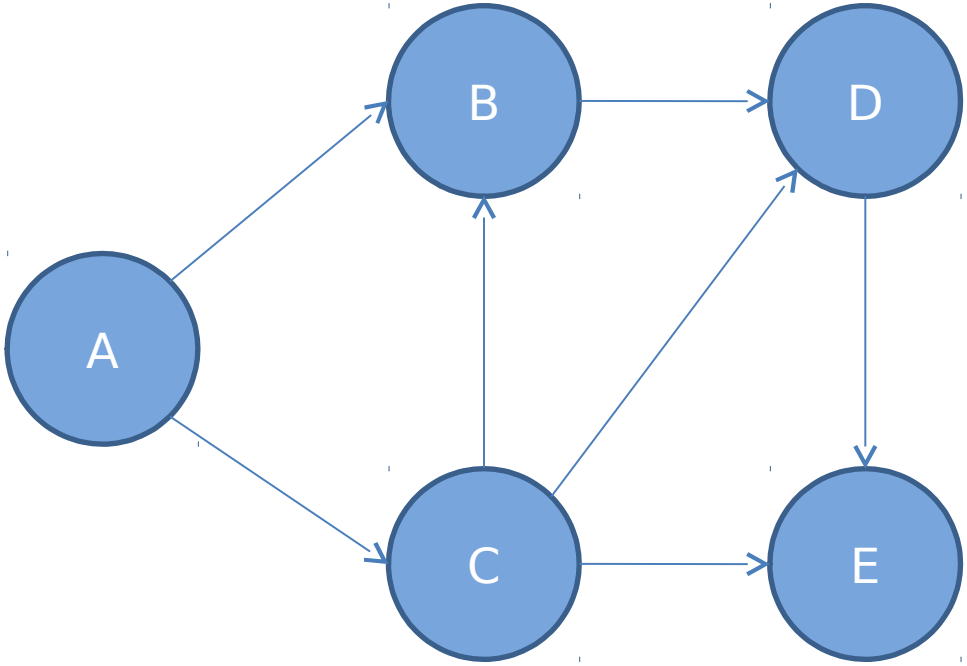
T-Sort:



Cycle Exists !!!

Depth First Search (DFS)

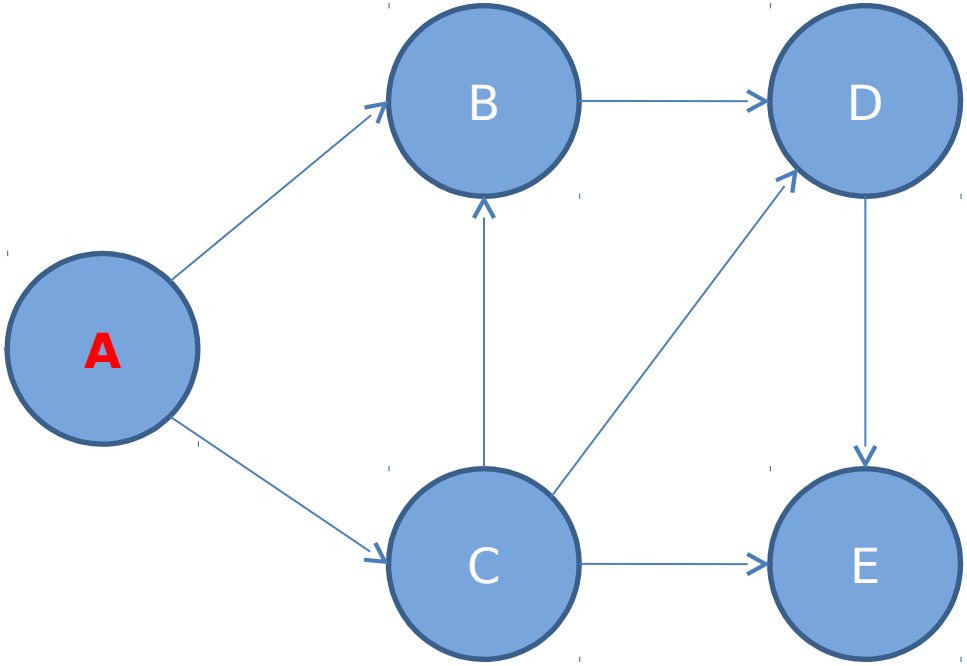
T-Sort:



OK

Depth First Search (DFS)

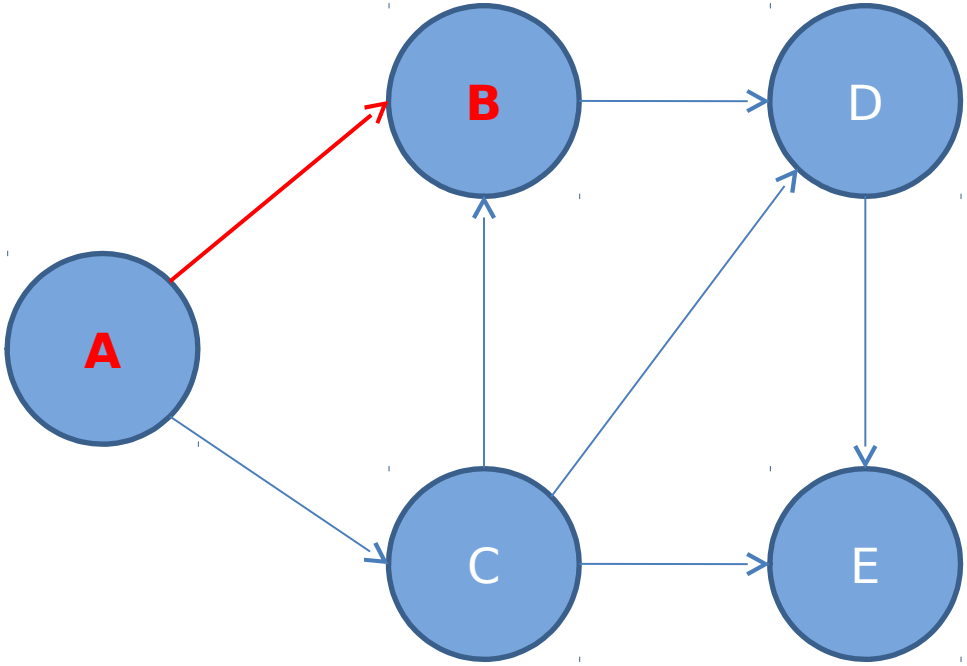
T-Sort:



Things in output stack:
NONE

Depth First Search (DFS)

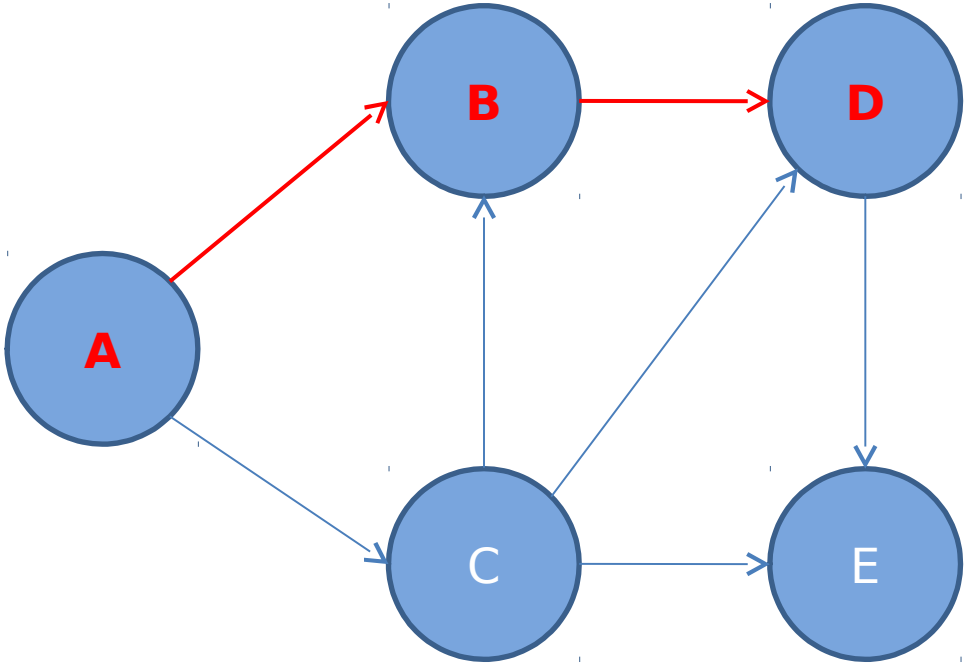
T-Sort:



Things in output stack:
NONE

Depth First Search (DFS)

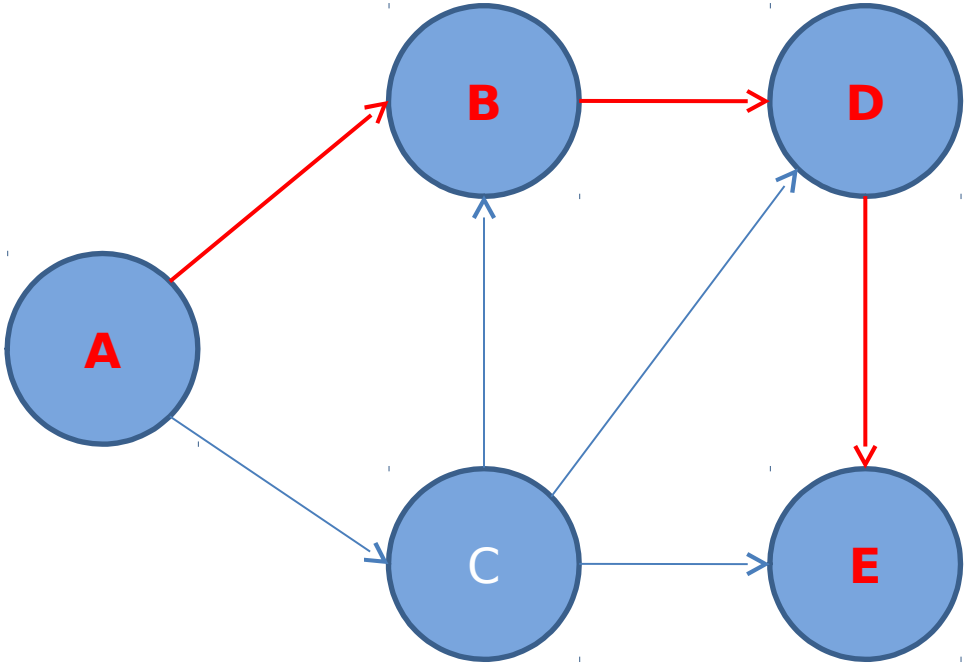
T-Sort:



Things in output stack:
NONE

Depth First Search (DFS)

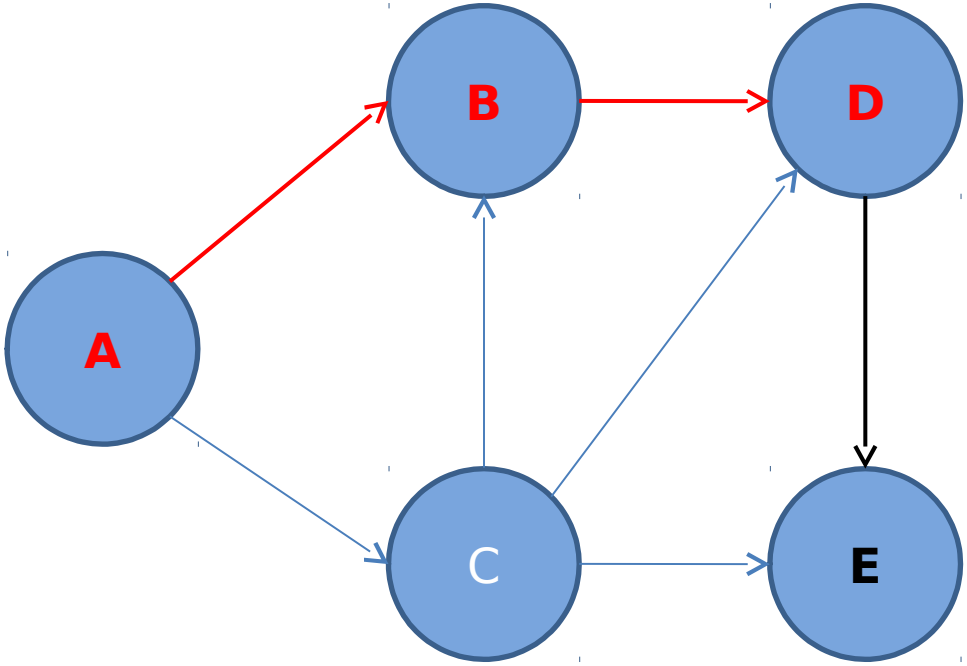
T-Sort:



Things in output stack:
NONE

Depth First Search (DFS)

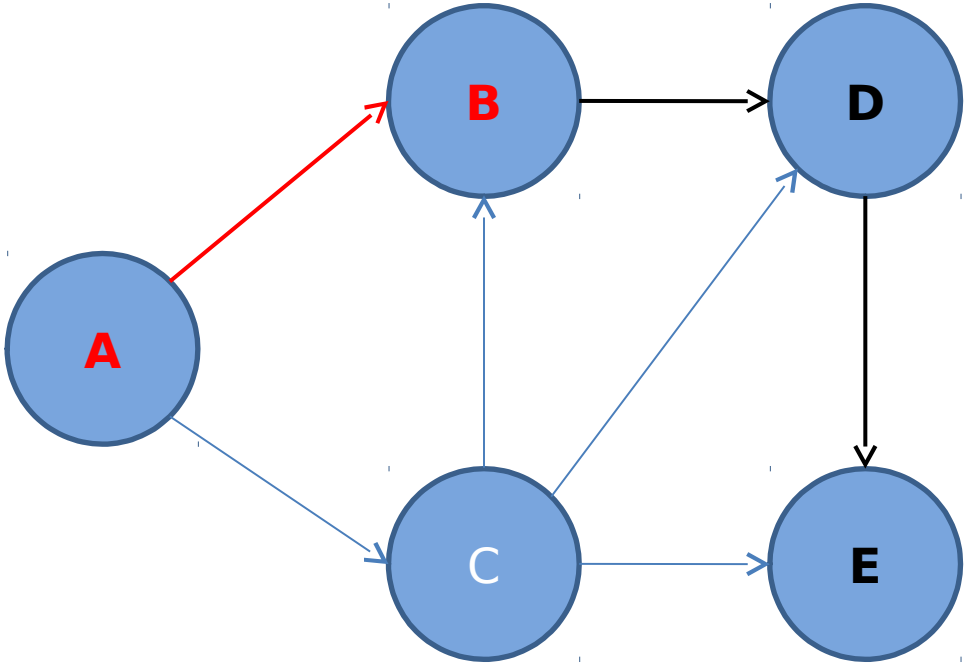
T-Sort:



Things in output stack: E

Depth First Search (DFS)

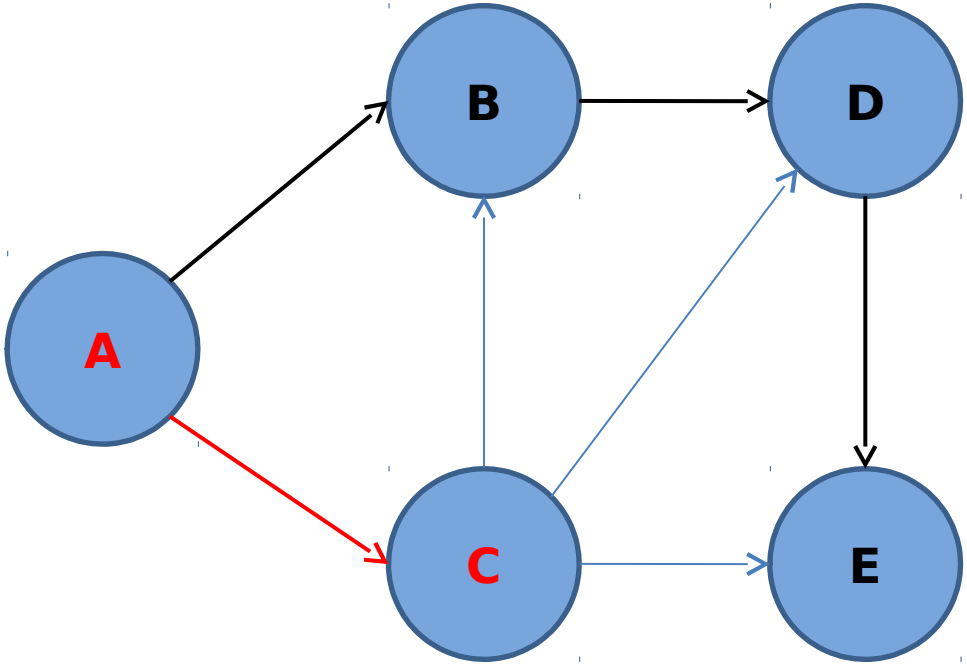
T-Sort:



Things in output stack: E, D

Depth First Search (DFS)

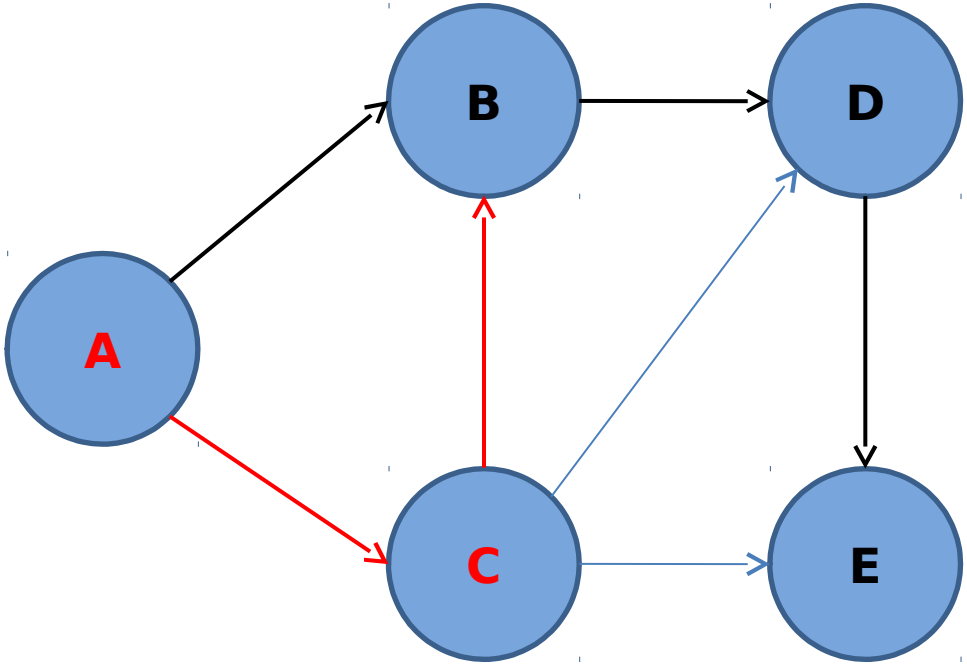
T-Sort:



Things in output stack: E, D, B

Depth First Search (DFS)

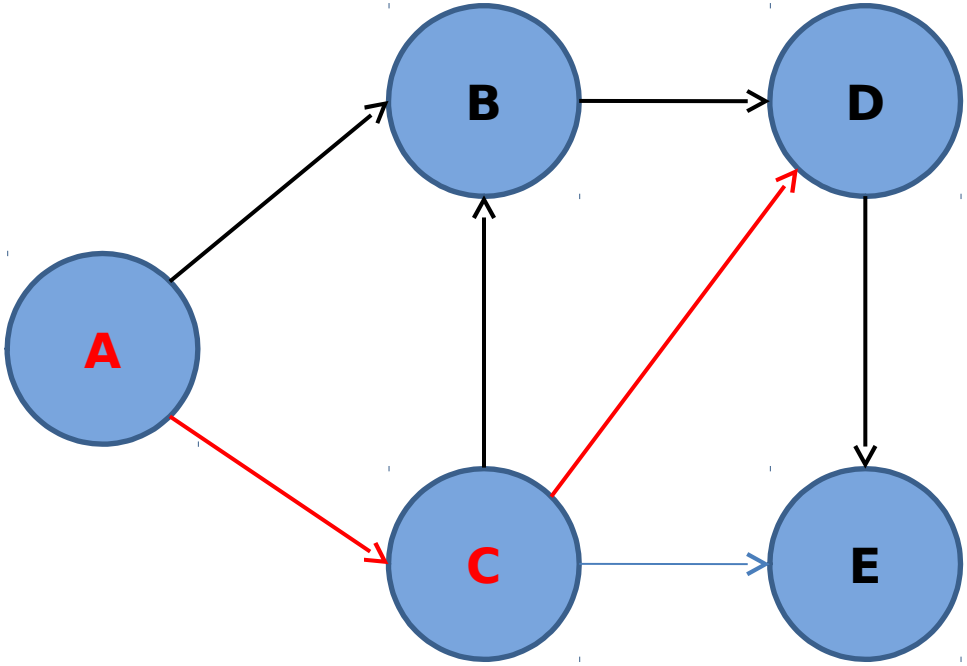
T-Sort:



Things in output stack: E, D, B

Depth First Search (DFS)

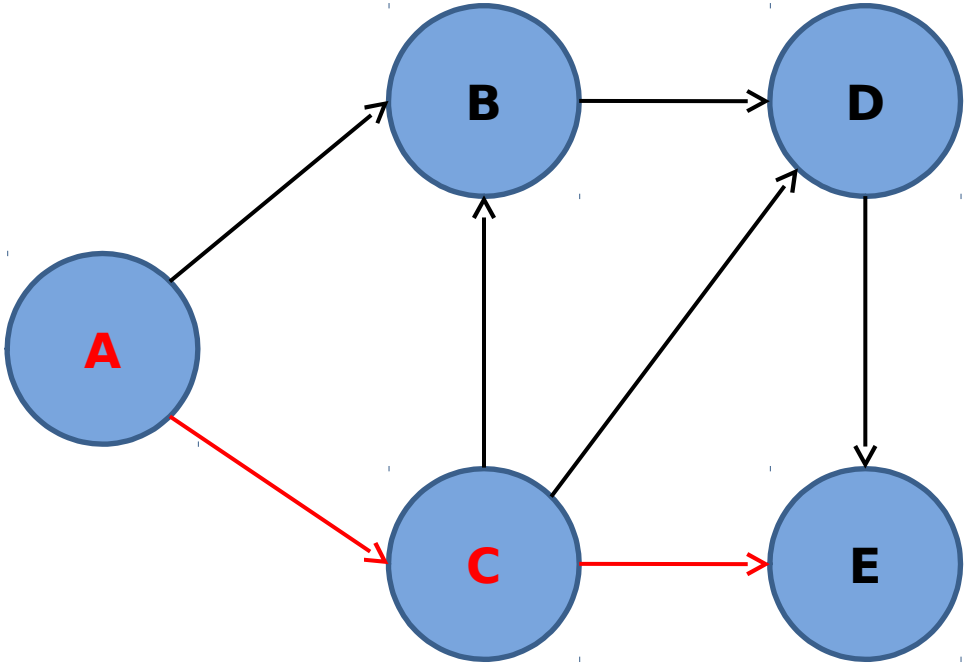
T-Sort:



Things in output stack: E, D, B

Depth First Search (DFS)

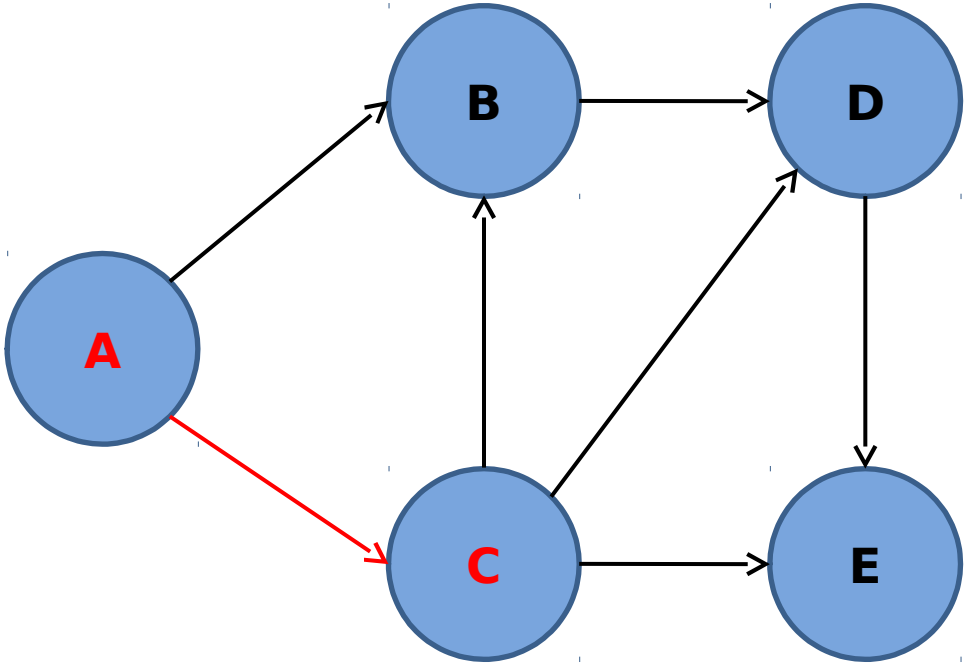
T-Sort:



Things in output stack: E, D, B

Depth First Search (DFS)

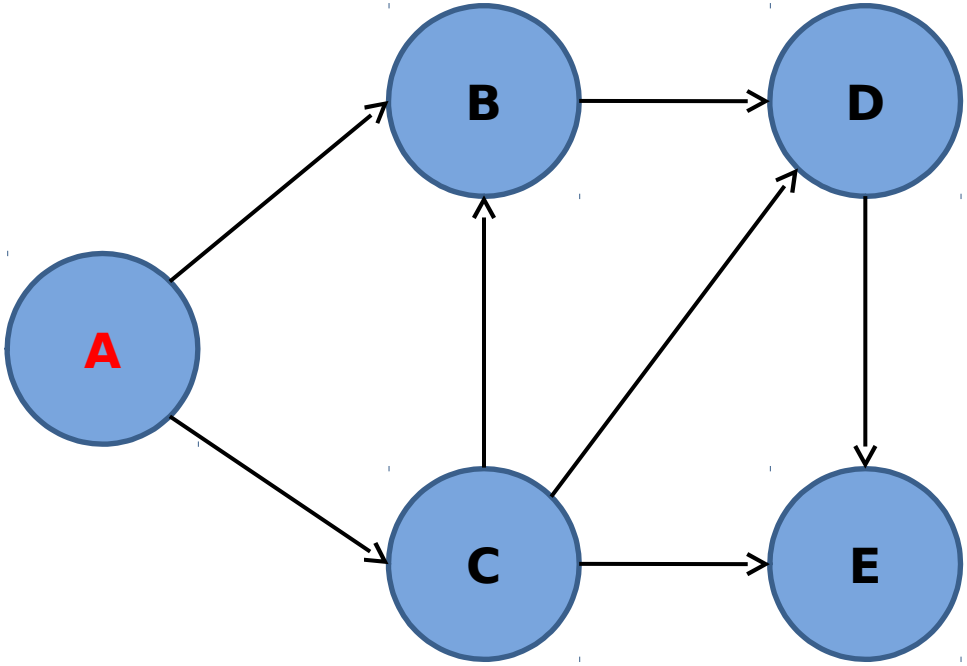
T-Sort:



Things in output stack: E, D, B

Depth First Search (DFS)

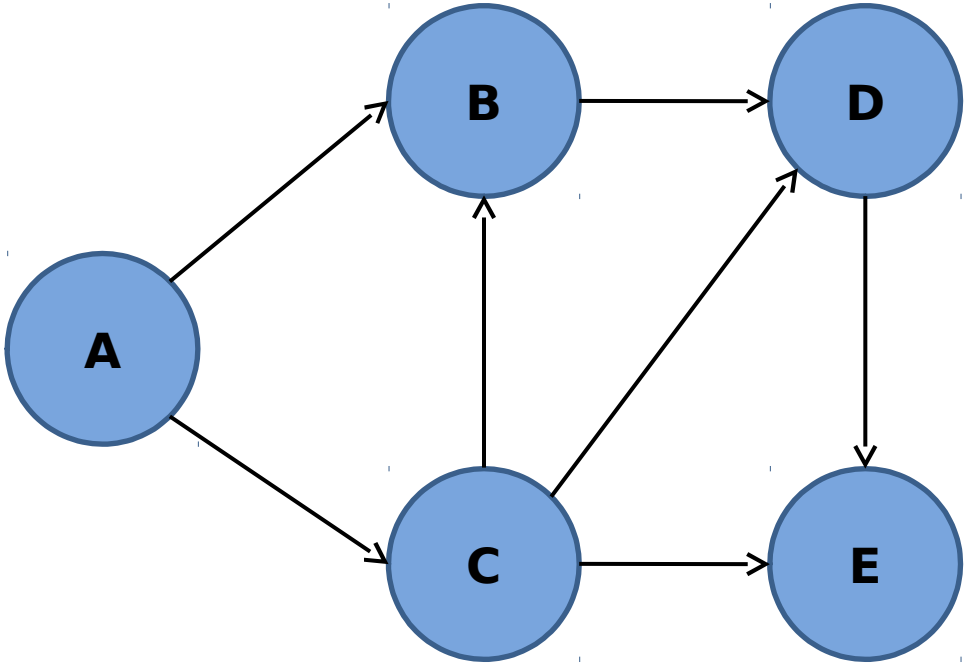
T-Sort:



Things in output stack: E, D, B, C

Depth First Search (DFS)

T-Sort:



Things in output stack: E, D, B, C, A
T-Sort Output: A, C, B, D, E

Depth First Search (DFS)

- **Strongly Connected Components**
  - **Directed Graph**
  - **Find groups of nodes such that in each group, every node has a path to every other node**

Depth First Search (DFS)

- **Strongly Connected Components**
  - **Method: DFS twice (Kosaraju)**
  - **Do DFS on graph G from all vertices, record finishing time of node**
  - **Reverse direction of edges**
  - **Do DFS on Graph G from all vertices sorted by decreasing finishing time**
  - **Each DFS tree is an SCC**

Depth First Search (DFS)

- **Strongly Connected Components**
- **Note: After grouping the vertices, the new nodes form a Directed Acyclic Graph**

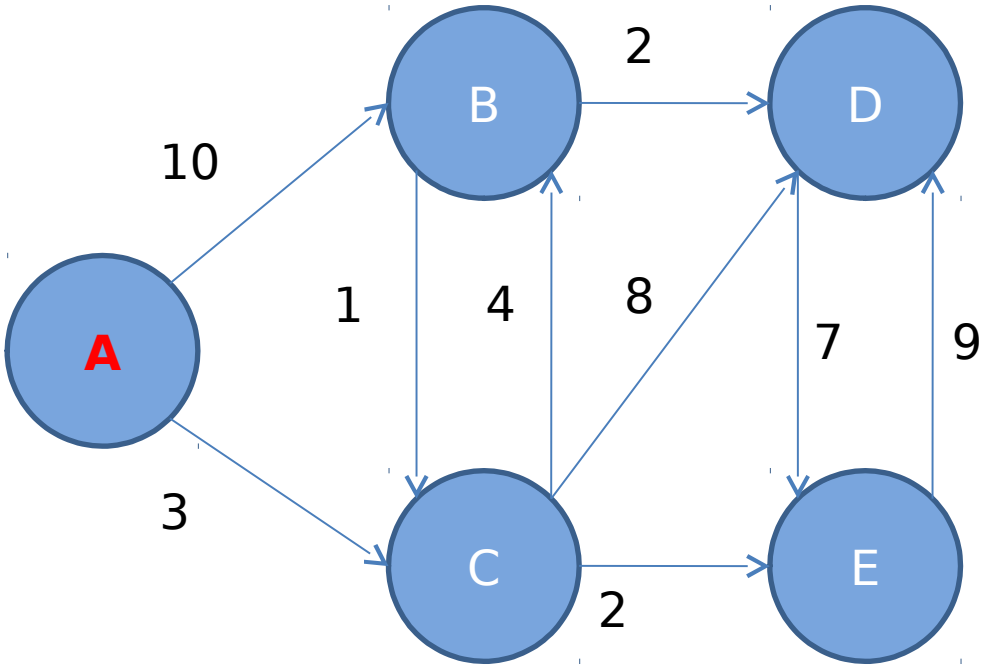
Breadth First Search (BFS)

- **Structure to use: Queue**

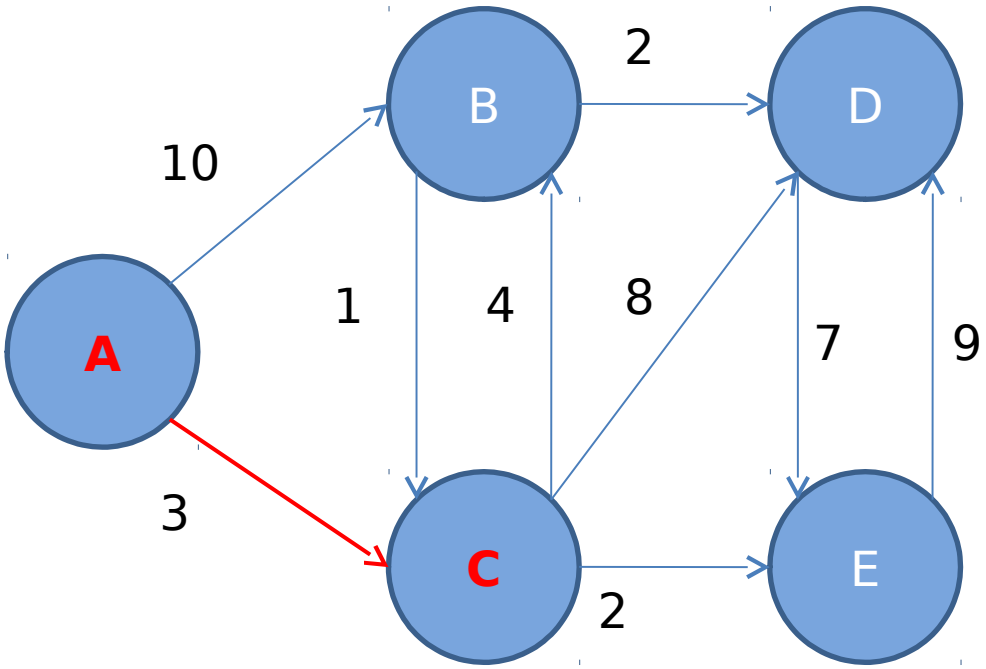
Breadth First Search (BFS)

Things in queue:

A, C	3
A, B	10



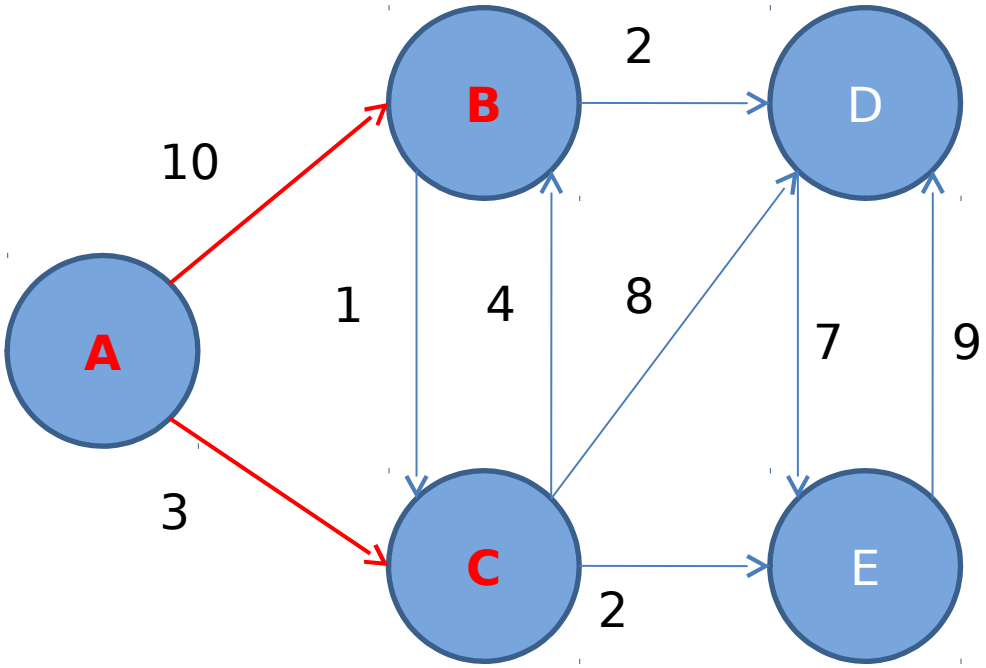
Breadth First Search (BFS)



Things in queue:

A, B	10
A, C, E	5
A, C, B	7
A, C, D	8

Breadth First Search (BFS)



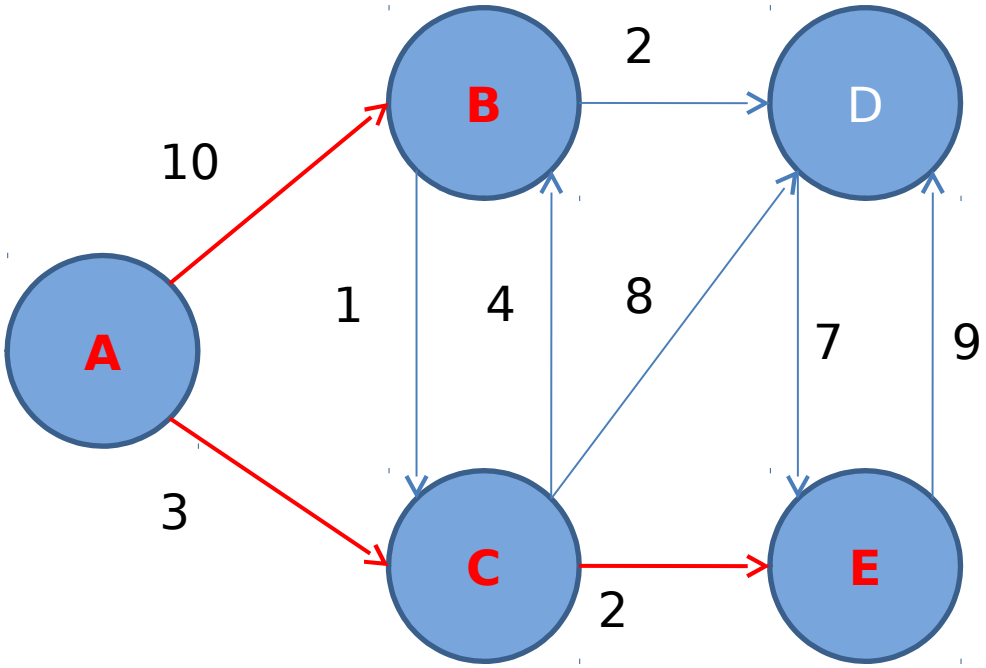
Things in queue:

A, C, E 5
A, C, B 7
A, C, D 8
A, B, C 11
A, B, D 12

Breadth First Search (BFS)

Things in queue:

A, C, B 7
A, C, D 8
A, B, C 11
A, B, D 12



We are done !!!

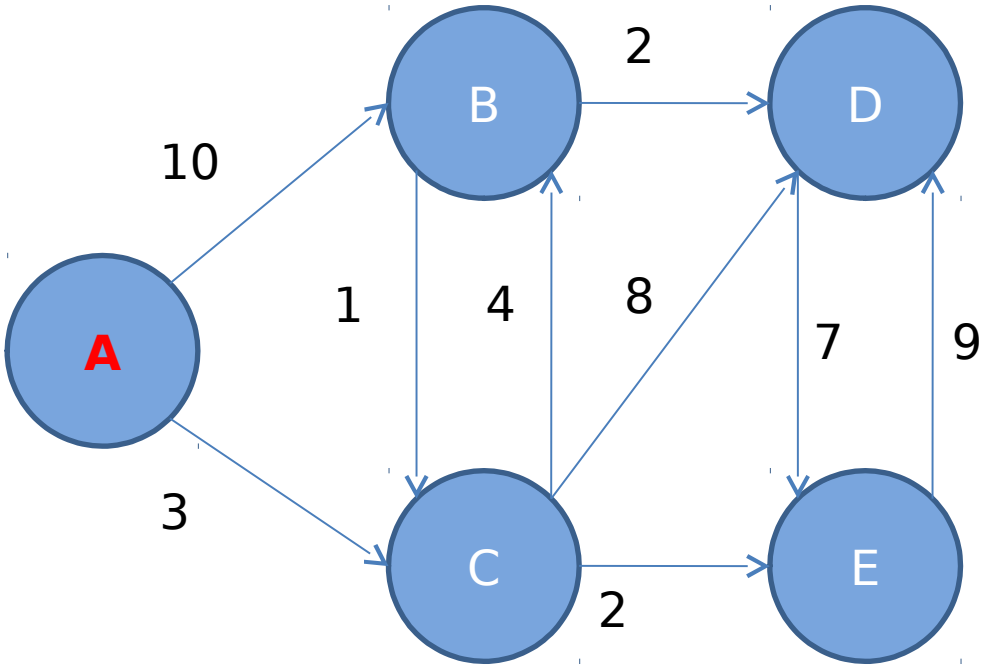
Breadth First Search (BFS)

- **Application:**
  - **Finding shortest path**
  - **Flood-Fill**

Dijkstra

- **Structure to use: Priority Queue**
- **Consider the past**
  - **The length of path visited so far**
- **No negative edges allowed**
- **Maintain known vertices**
- **At each step:**
  - **Take the unknown vertex with smallest overall estimate**

Dijkstra



Things in priority queue:

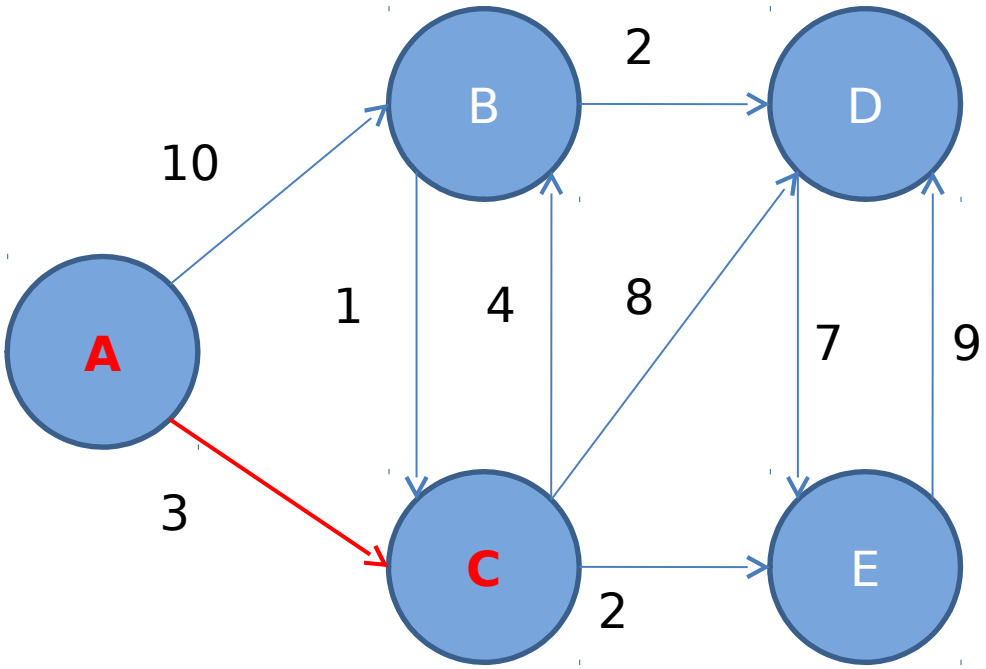
A, C 3

A, B 10

Current Route:

A

Dijkstra



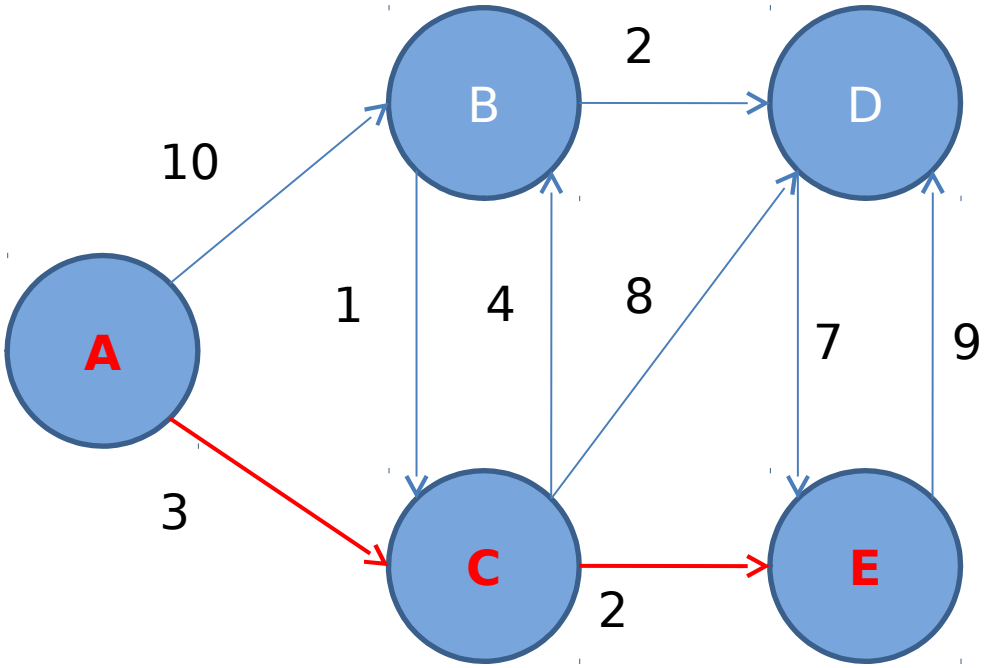
Things in priority queue:

A, C, E 5
A, C, B 7
A, B 10
A, C, D 11

Current Route:

A, C

Dijkstra



Things in priority queue:

A, C, B 7
A, B 10
A, C, D 11

Current Route:

A, C, E

We find the shortest path already !!!

A* Search

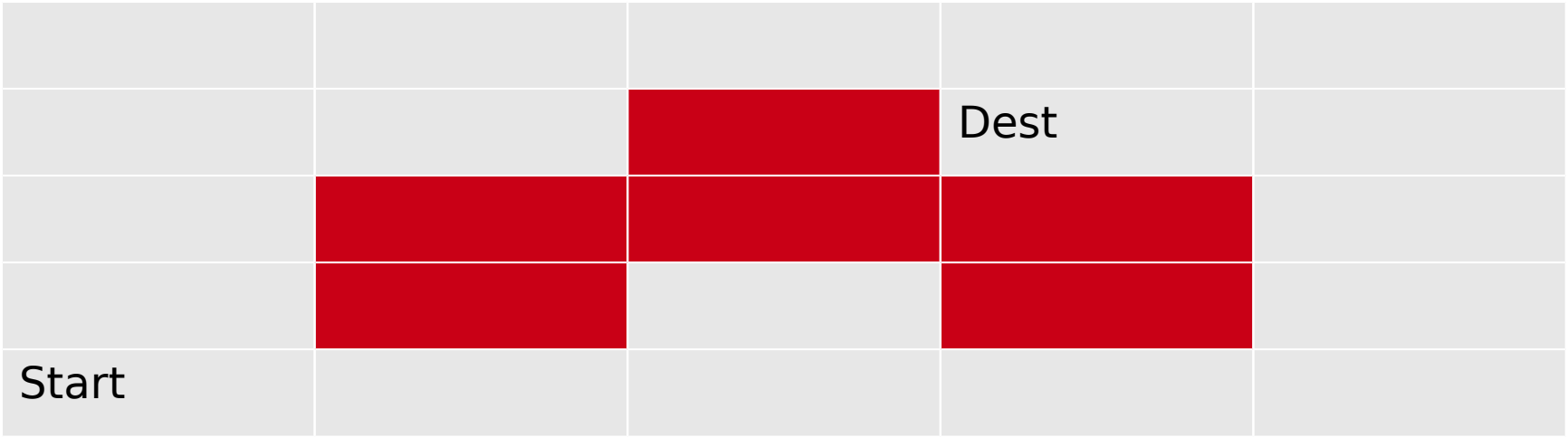
- **Structure to use: Priority Queue**
- **Consider the past + future**
- **How to consider the future?**
  - **Admissible Heuristic**

    (Best-Case Prediction: must never overestimate)
  - **How to predict?**

A* Search

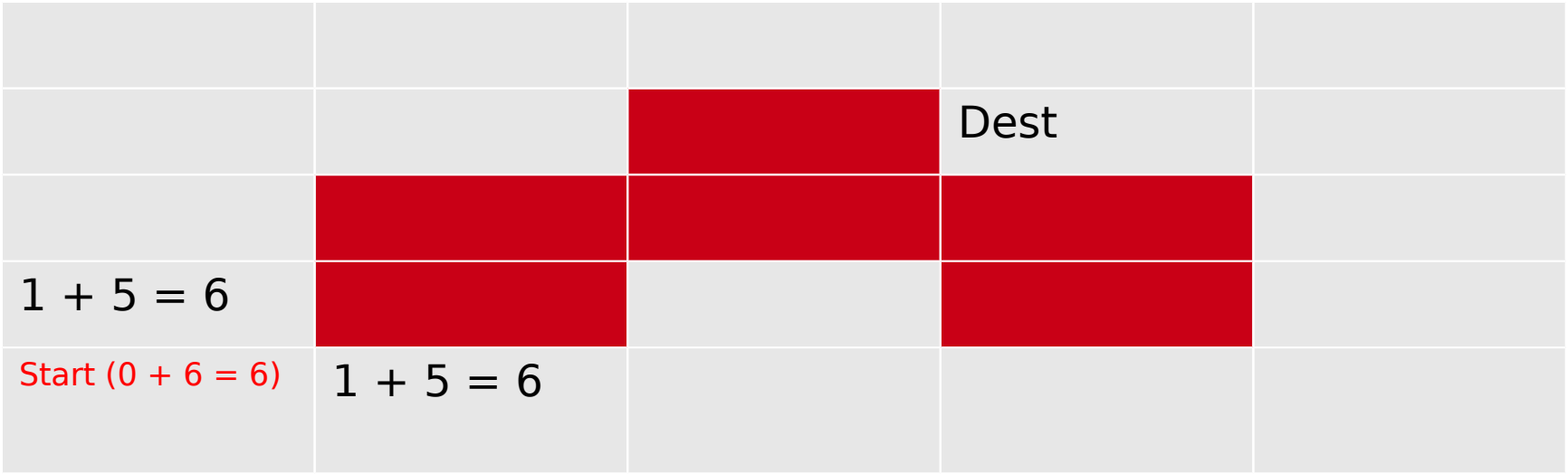
- **Prediction 1: Displacement**
  - **Given coordinates, calculate the displacement of current node to destination**
  - **$\text{sqrt}((x2 - x1)^2 + (y2 - y1)^2)$**
- **Prediction 2: Manhattan Distance**
  - **Given grid, calculate the horizontal and vertical movement**
  - **(x2 – x1) + (y2 – y1)**

A* Search



- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search



- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

			Dest	
2 + 4 = 6				
1 + 5 = 6				
Start (0 + 6 = 6)	1 + 5 = 6			

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

3 + 3 = 6			Dest	
2 + 4 = 6				
1 + 5 = 6				
Start (0 + 6 = 6)	1 + 5 = 6			

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

4 + 4 = 8				
3 + 3 = 6	4 + 2 = 6		Dest	
2 + 4 = 6				
1 + 5 = 6				
Start (0 + 6 = 6)	1 + 5 = 6			

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

4 + 4 = 8	5 + 3 = 8			
3 + 3 = 6	4 + 2 = 6		Dest	
2 + 4 = 6				
1 + 5 = 6				
Start (0 + 6 = 6)	1 + 5 = 6			

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

4 + 4 = 8	5 + 3 = 8			
3 + 3 = 6	4 + 2 = 6		Dest	
2 + 4 = 6				
1 + 5 = 6				
Start (0 + 6 = 6)	1 + 5 = 6	2 + 4 = 6		

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

4 + 4 = 8	5 + 3 = 8			
3 + 3 = 6	4 + 2 = 6		Dest	
2 + 4 = 6				
1 + 5 = 6		3 + 3 = 6		
Start (0 + 6 = 6)	1 + 5 = 6	2 + 4 = 6	3 + 3 = 6	

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

4 + 4 = 8	5 + 3 = 8			
3 + 3 = 6	4 + 2 = 6		Dest	
2 + 4 = 6				
1 + 5 = 6		3 + 3 = 6		
Start (0 + 6 = 6)	1 + 5 = 6	2 + 4 = 6	3 + 3 = 6	

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

4 + 4 = 8	5 + 3 = 8			
3 + 3 = 6	4 + 2 = 6		Dest	
2 + 4 = 6				
1 + 5 = 6		3 + 3 = 6		
Start (0 + 6 = 6)	1 + 5 = 6	2 + 4 = 6	3 + 3 = 6	4 + 4 = 8

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

4 + 4 = 8	5 + 3 = 8	6 + 2 = 8		
3 + 3 = 6	4 + 2 = 6		Dest	
2 + 4 = 6				
1 + 5 = 6		3 + 3 = 6		
Start (0 + 6 = 6)	1 + 5 = 6	2 + 4 = 6	3 + 3 = 6	4 + 4 = 8

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

4 + 4 = 8	5 + 3 = 8	6 + 2 = 8	7 + 1 = 8	
3 + 3 = 6	4 + 2 = 6		Dest	
2 + 4 = 6				
1 + 5 = 6		3 + 3 = 6		
Start (0 + 6 = 6)	1 + 5 = 6	2 + 4 = 6	3 + 3 = 6	4 + 4 = 8

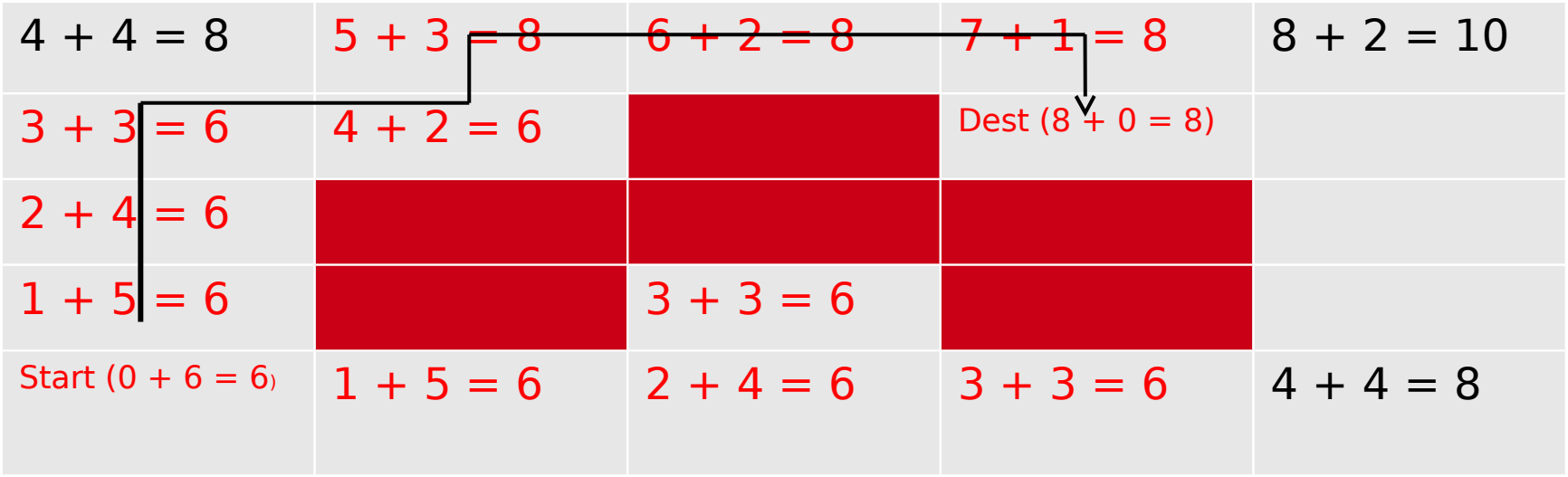
- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

4 + 4 = 8	5 + 3 = 8	6 + 2 = 8	7 + 1 = 8	8 + 2 = 10
3 + 3 = 6	4 + 2 = 6		Dest (8 + 0 = 8)	
2 + 4 = 6				
1 + 5 = 6		3 + 3 = 6		
Start (0 + 6 = 6)	1 + 5 = 6	2 + 4 = 6	3 + 3 = 6	4 + 4 = 8

- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search



- Manhattan Distance is used
- If there is a tie, consider the one with lowest heuristic first
- If there is still a tie, consider in Up, Down, Left, Right order

A* Search

- **A* Search = Past + Future**
- **A* Search – Future = ?**
  Dijkstra
- **A* Search – Past = ?**
  Greedy

Graph Theory

- **How about graph with negative edges?**
  - **Bellman-Ford**

Bellman-Ford Algorithm

- **Consider the source as weight 0, others as infinity**
- **Count = 0, Improve = true**
- **While (Count < n and Improve) {**
  - **Improve = false**
  - **Count++**
  - **For each edge uv**
    - **If u.weight + uv distance < v.weight {**
      - **Improve = true**
      - **v.weight = u.weight + uv.distance**
    - **}**
- **}**
- **If (Count == n)**
  print “Negative weight cycle detected”
- **Else print shortest path distance**

Bellman-Ford Algorithm

- **Time Compleity: O(VE)**

Graph Theory

- **All we have just dealt with is single source**
- **How about multiple sources (all sources)?**
  - **Floyd-Warshall Algorithm**

Floyd-Warshall Algorithm

- **1. Compute on subgraph {1}**
- **2. Compute on subgraph {1 ,2}**
- **3. Compute on subgraph {1, …, 3}**
- **4. …**
- **N. Compute on entire graph**
- **Done !**

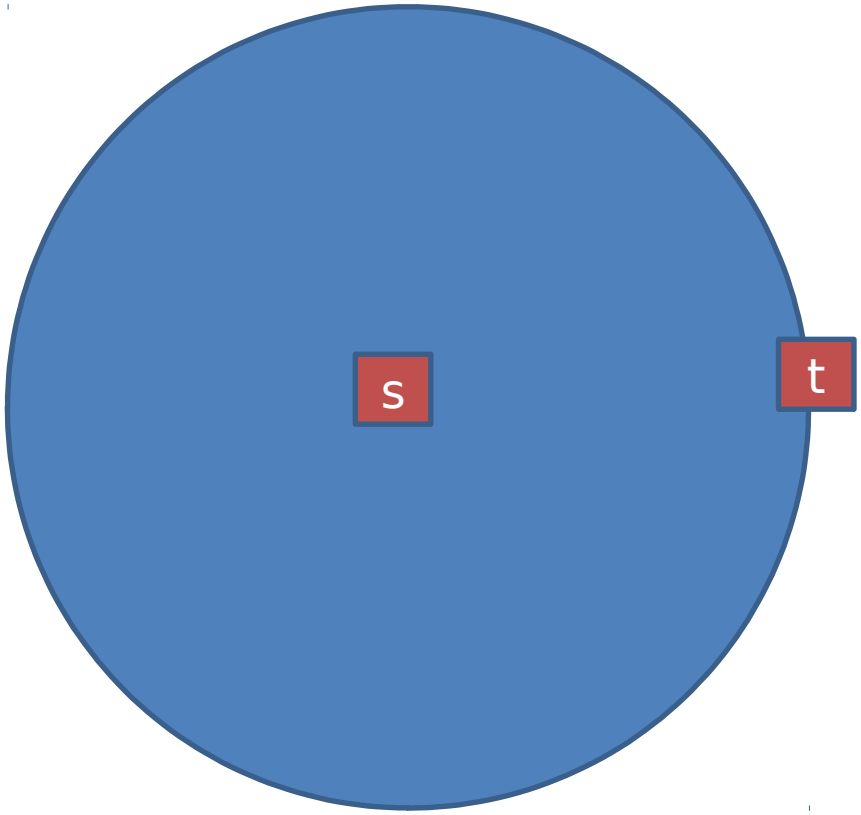
Floyd-Warshall Algorithm

- **For (int k = 0; k < MAX; k++) {**
  - **For (int i = 0; i < MAX; i++) {**
    - **For (int j = 0; j < MAX; j++) {**
      - **Path[i][j] = min(Path[i][j], Path[i][k] + Path[k][j]);**
    - **}**
  - **}**
- **}**

Iterative? Non-iterative?

- **What is iterative-deepening?**
- **Given limited time, f nd the current most optimal solution**
- **Dijkstra (Shortest 1-step Solution)**
- **Dijkstra (Shortest 2-steps Solution)**
- **Dijkstra (Shortest 3-steps Solution)**
- **…**
- **Dijkstra (Shortest n-steps Solution)**

- **Suitable in bi-directional search (Faster than 1-way search): only for DFS, tricky otherwise**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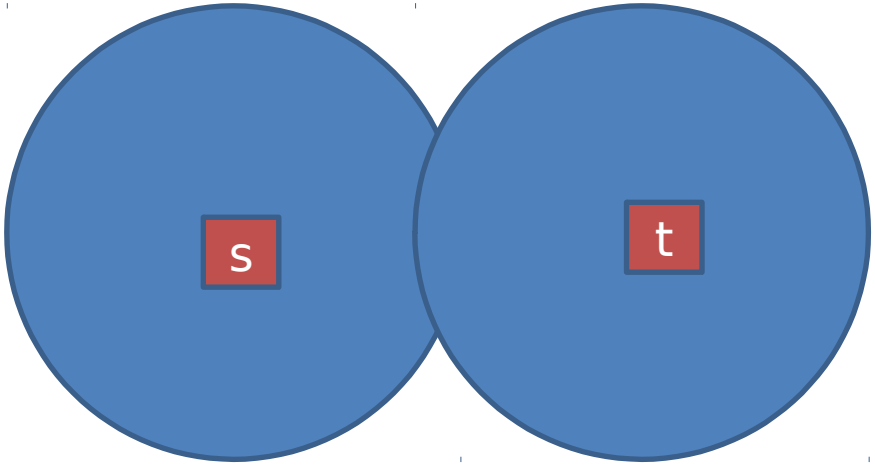
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Minimum Spanning Tree (MST)

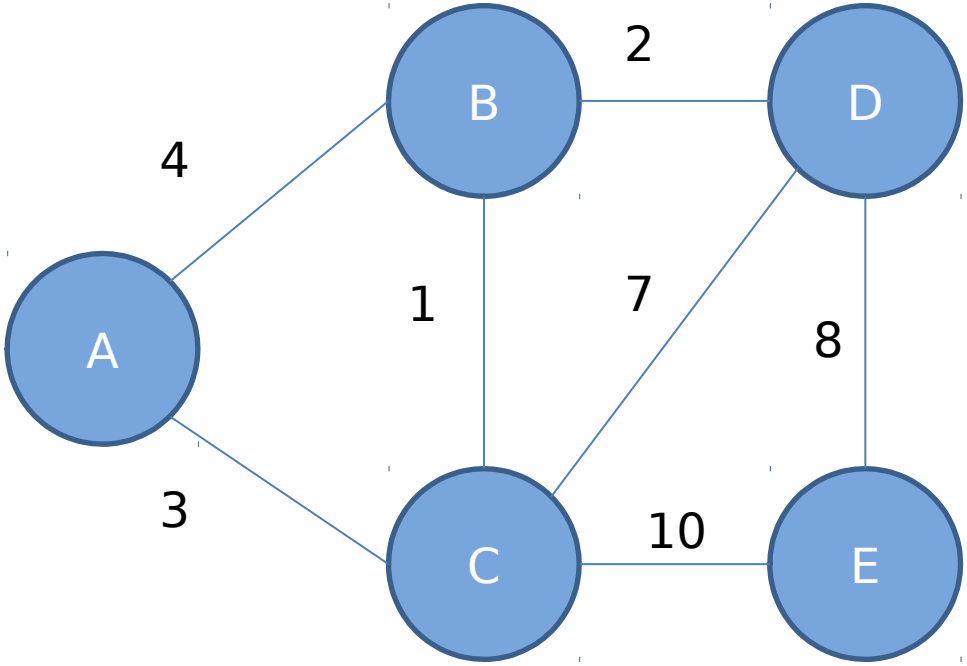
- **Given an undirected graph, find the minimum cost so that every node has path to other nodes**

- **Kruskal’s Algorithm**
- **Prim’s Algorithm**

Kruskal's Algorithm

- **Continue Finding Shortest Edge**
- **If no cycle is formed**
  - **add it into the list**
  - **Construct a parent-child relationship**
- **If all nodes have the same root, we are done (path compression)**
- **General idea: union-find, may be useful in other situations**

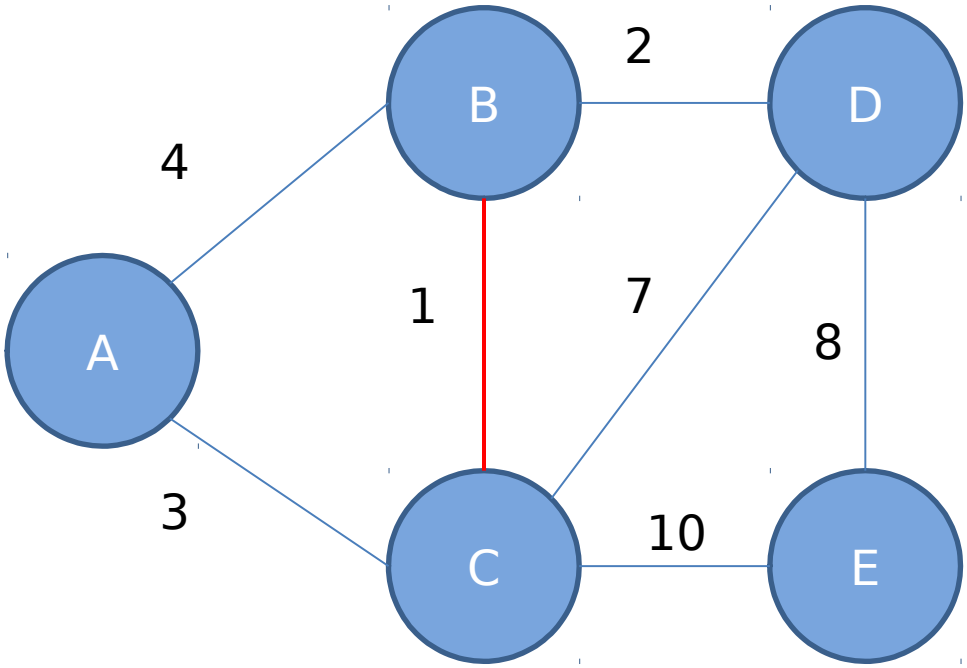
Kruskal’s Algorithm



Node	A	B	C	D	E
Parent	A	B	C	D	E

Kruskal's Algorithm

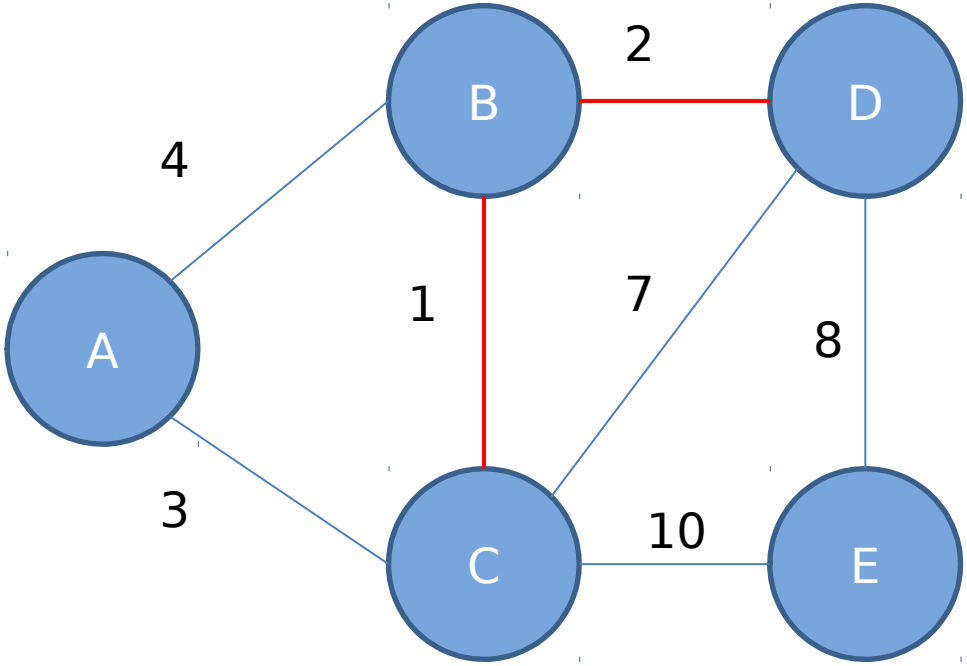
Edge List:
BC 1



Node	A	B	C	D	E
Parent	A	B	B	D	E

Kruskal's Algorithm

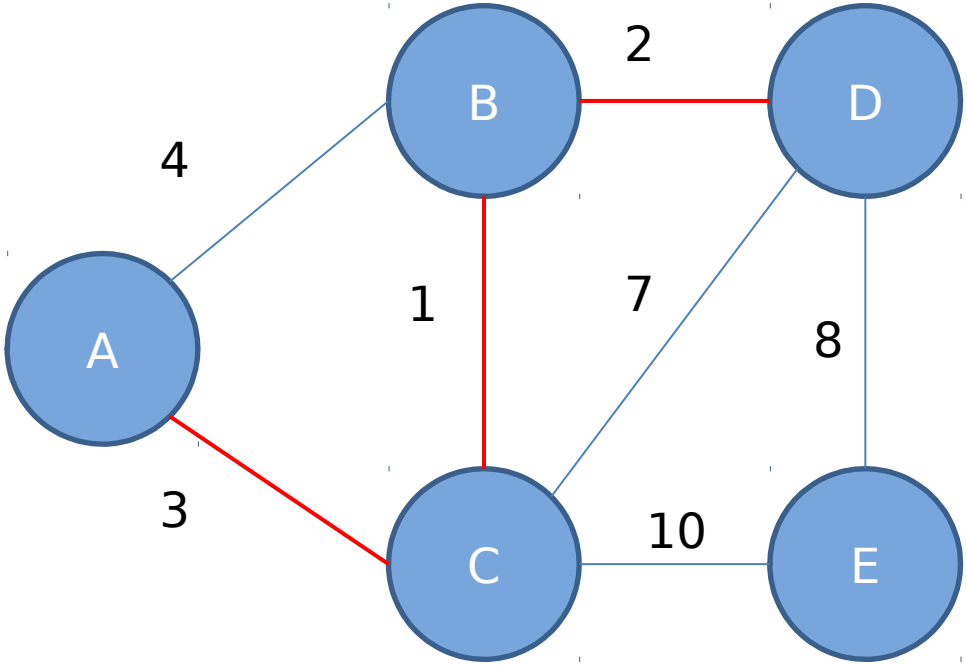
Edge List:
BC 1



Node	A	B	C	D	E
Parent	A	B	B	B	E

Kruskal's Algorithm

Edge List:
BC 1

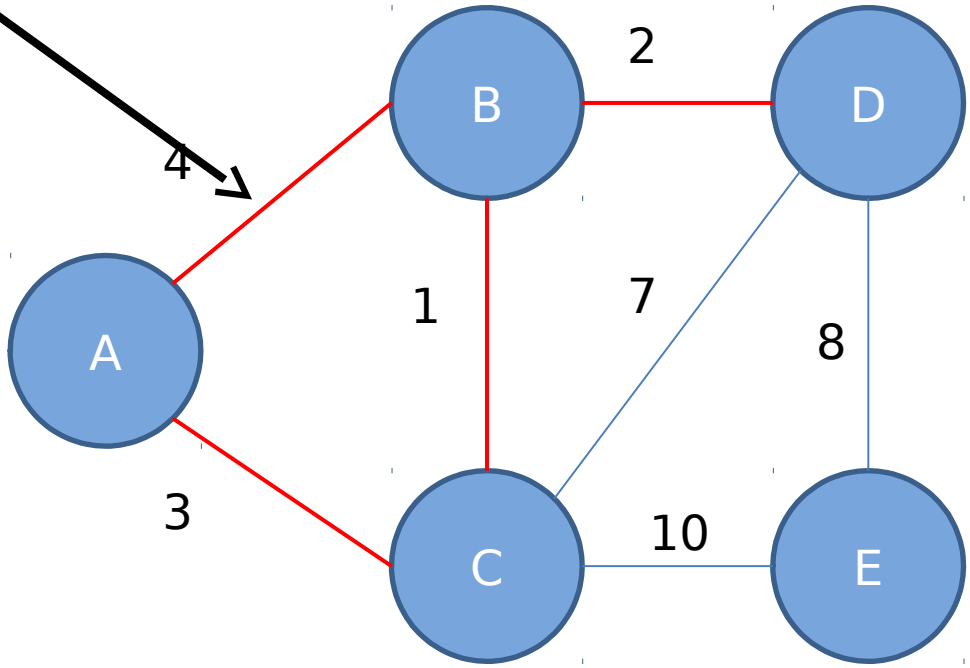


Node	A	B	C	D	E
Parent	A	A	A	A	E

Kruskal's Algorithm

Edge List:
BC 1

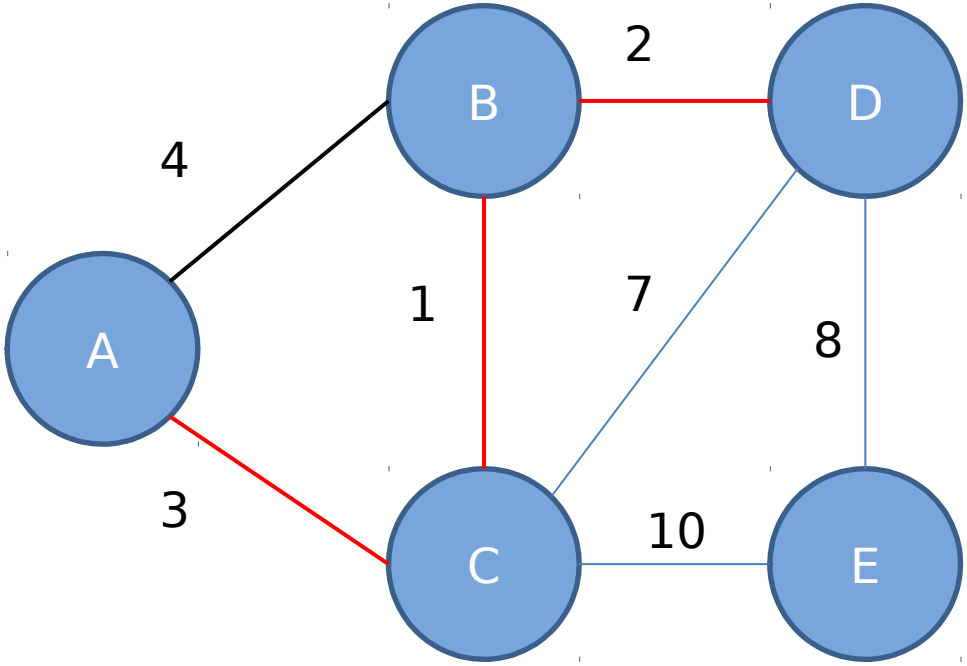
Cycle formed



Node	A	B	C	D	E
Parent	A	A	A	A	E

Kruskal's Algorithm

Edge List:
BC 1

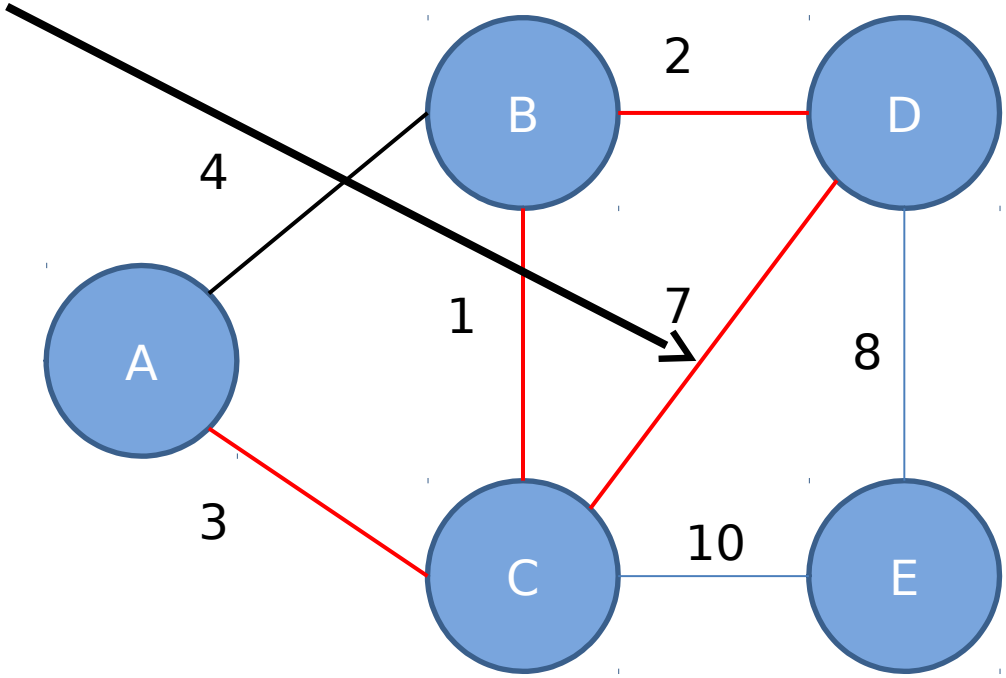


Node	A	B	C	D	E
Parent	A	A	A	A	E

Kruskal's Algorithm

Edge List:
BC 1

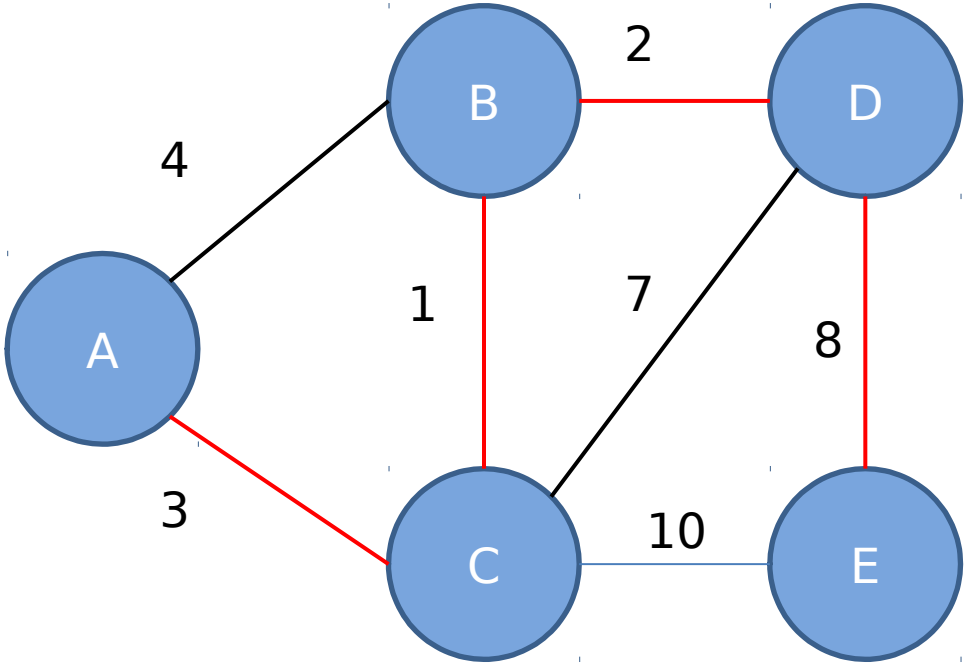
Cycle formed



Node	A	B	C	D	E
Parent	A	A	A	A	E

Kruskal's Algorithm

Edge List:
BC 1



Node	A	B	C	D	E
Parent	A	A	A	A	A

Kruskal’s Algorithm

- **Path Compression can be done when we find root**

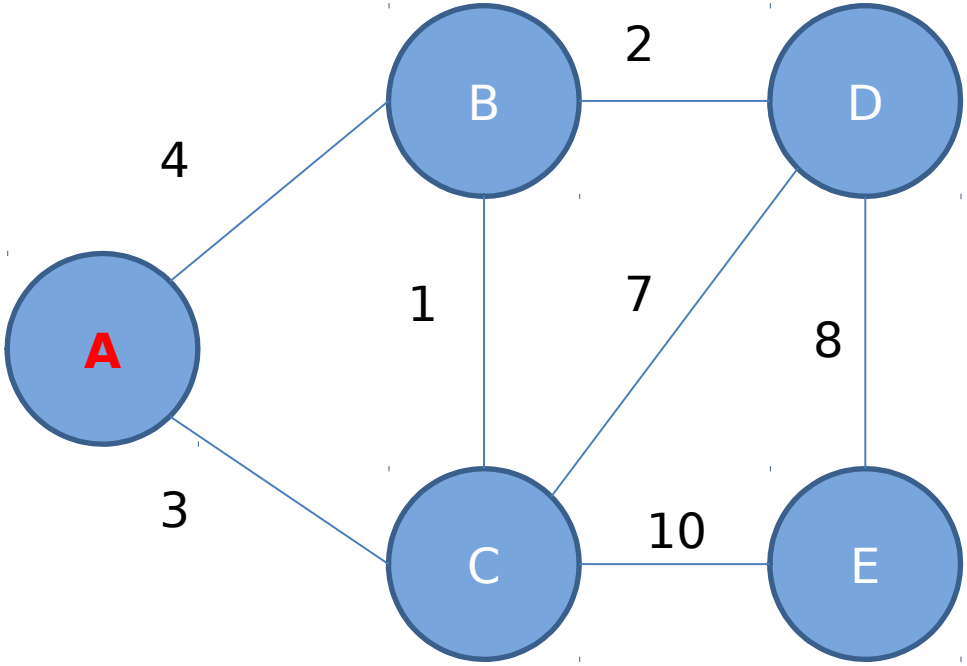
```
int find_root(int x) {
    return (parent[x] == x)? x:
    return (parent[x] = find_root(parent[x]));
}
```

Prim’s Algorithm

- **While not all nodes are visited**
  - **While the list is not empty and the minimum edge connects to visited node**
    - **Pop out the edge**
  - **If the list is empty, print Impossible and return**
  - **Else**
    - **Consider popped edge**
    - **Mark its endpoint as visited**
    - **Add all its unvisited edges to the list**
- **Print the tree**

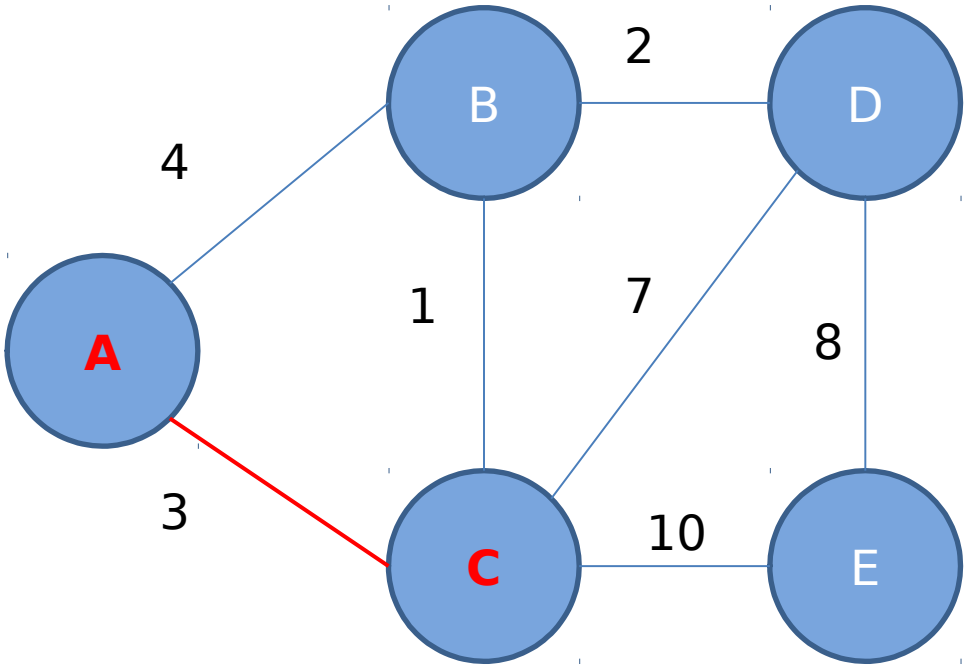
Prim’s Algorithm

Edge List:
AC 3
AB 4



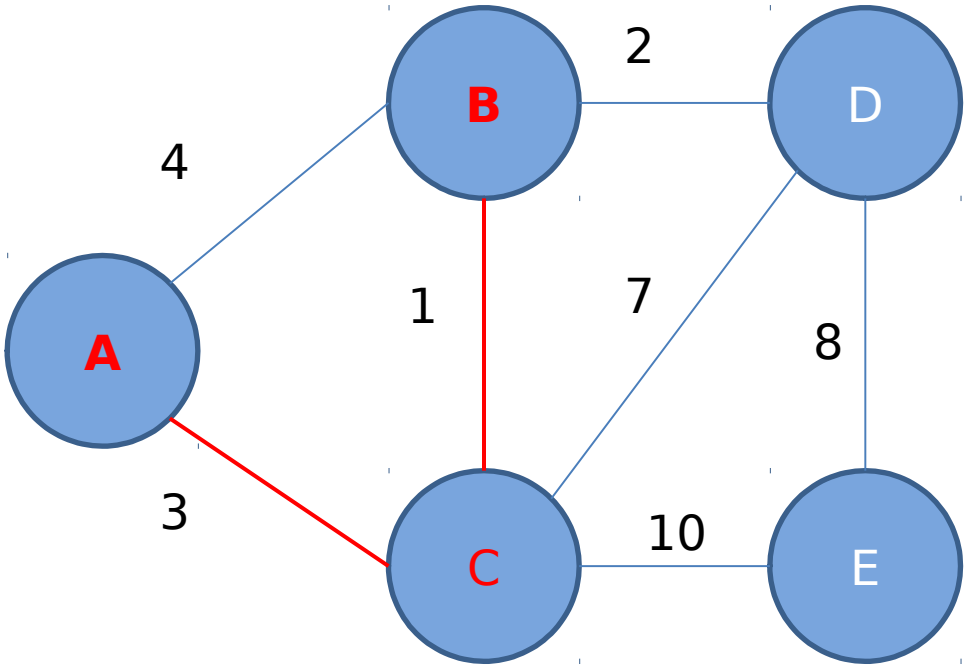
Prim’s Algorithm

Edge List:
CB 1
AB 4
CD 7
CE 10



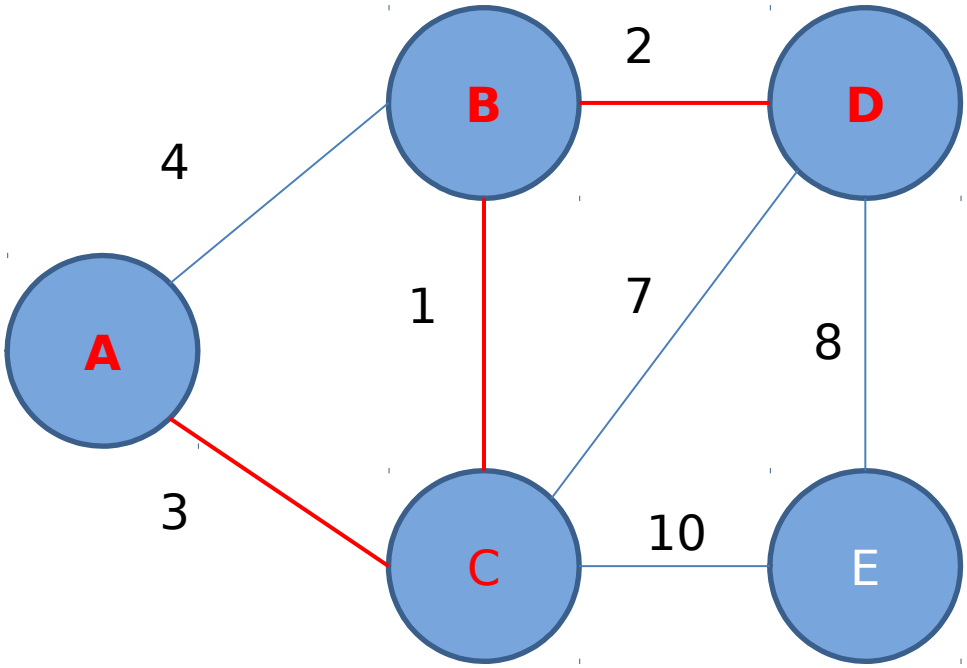
Prim’s Algorithm

Edge List:
BD 2
AB 4
CD 7
CE 10



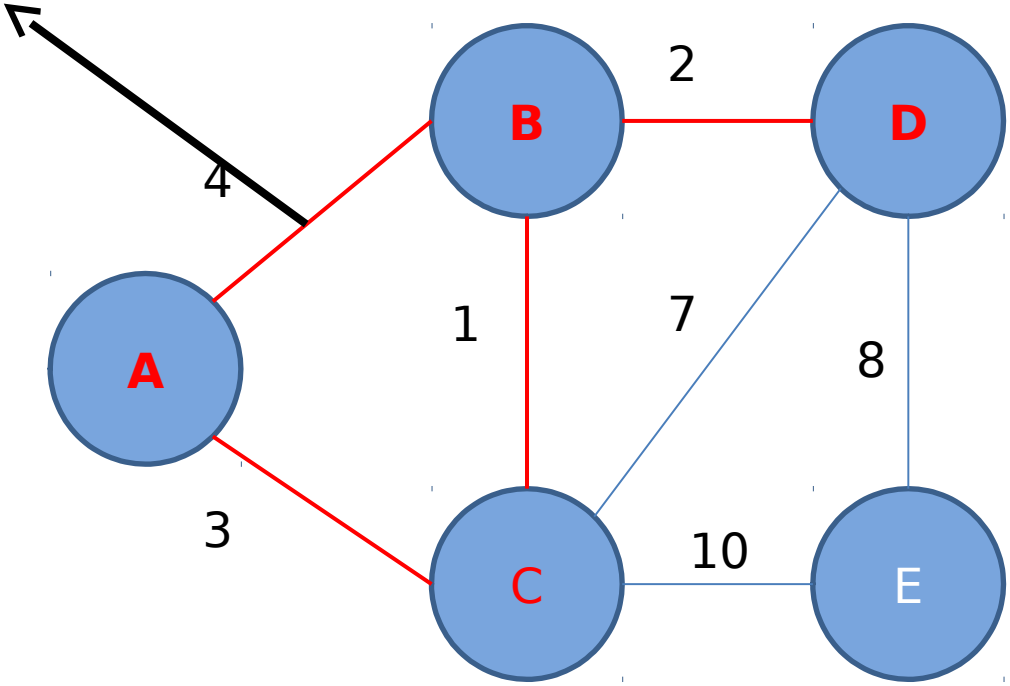
Prim’s Algorithm

Edge List:
AB 4
CD 7
DE 8
CE 10



Prim’s Algorithm

Cycle formed



Edge List:

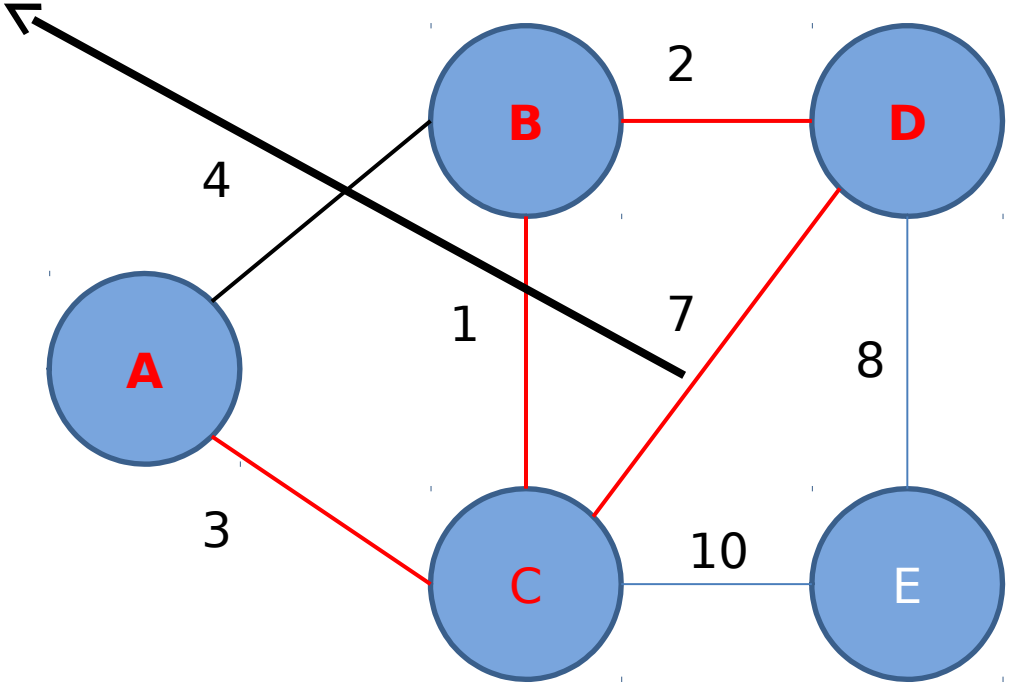
CD 7

DE 8

CE 10

Prim’s Algorithm

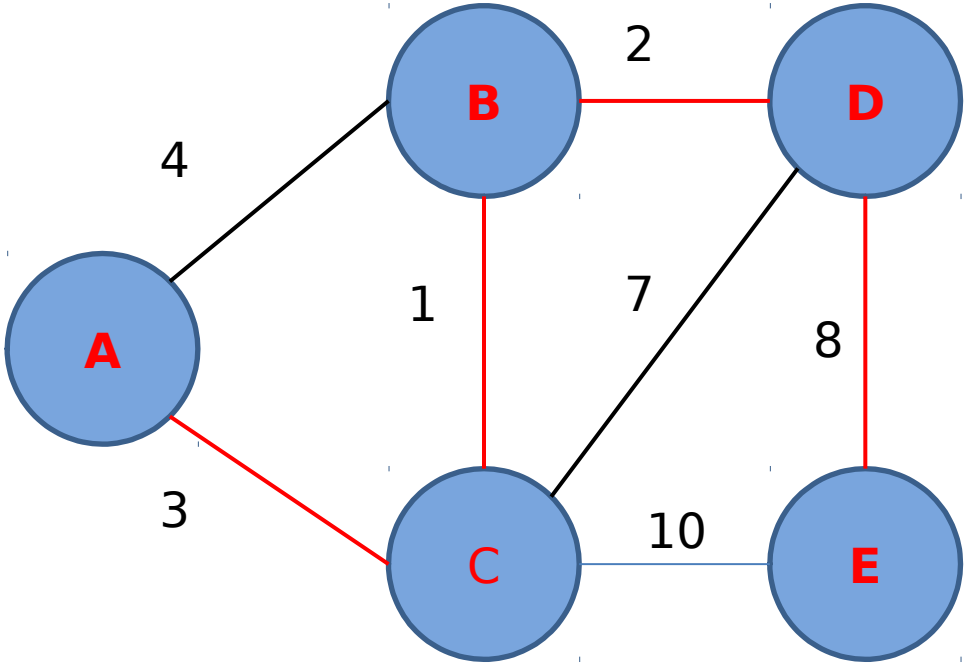
Cycle formed



Edge List:
DE 8
CE 10

Prim’s Algorithm

Edge List:
CE 10



We are done

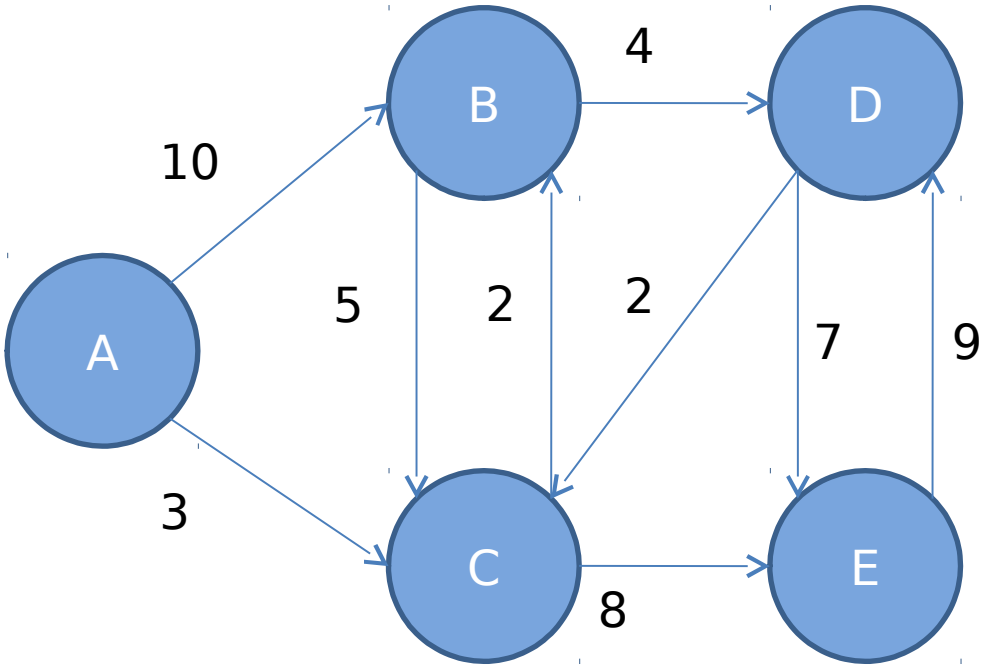
Directed Minimum Spanning Tree

- **We have just dealt with undirected MST.**
- **How about directed?**
- **Using Kruskal's or Prim's cannot solve directed MST problem**
- **How?**
- **Chu-Liu/Edmonds Algorithm**

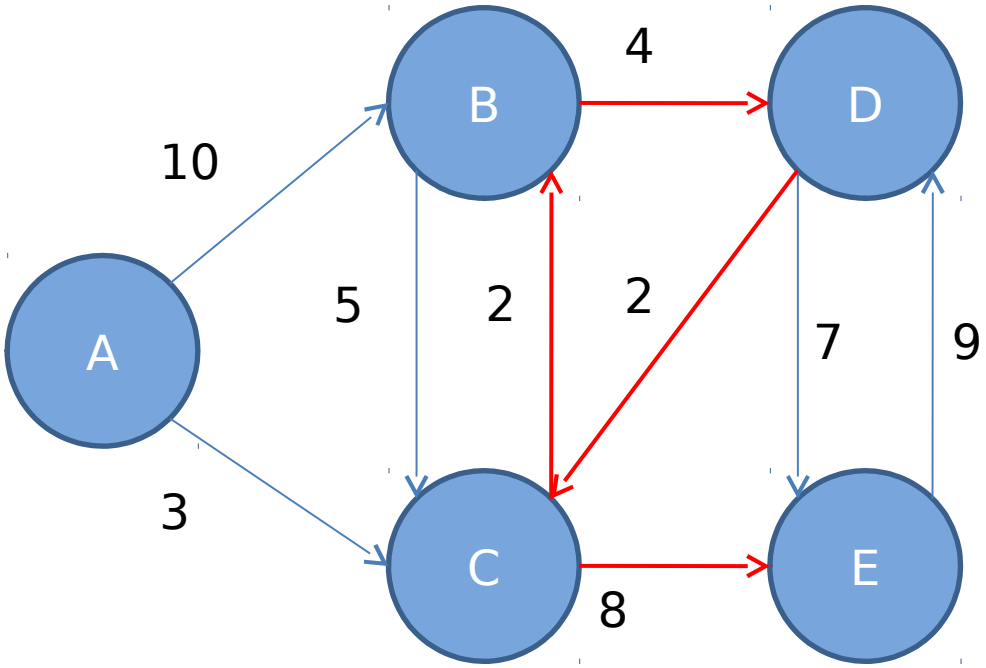
Chu-Liu/Edmonds Algorithm

- **For each vertex, find the smallest incoming edge**
- **While there is cycle**
  - **Find an edge entering the cycle with smallest increase**
  - **Remove the edge pointing to the same node and replace by new edge**
- **Print out the tree**

Chu-Liu/Edmonds Algorithm



Chu-Liu/Edmonds Algorithm



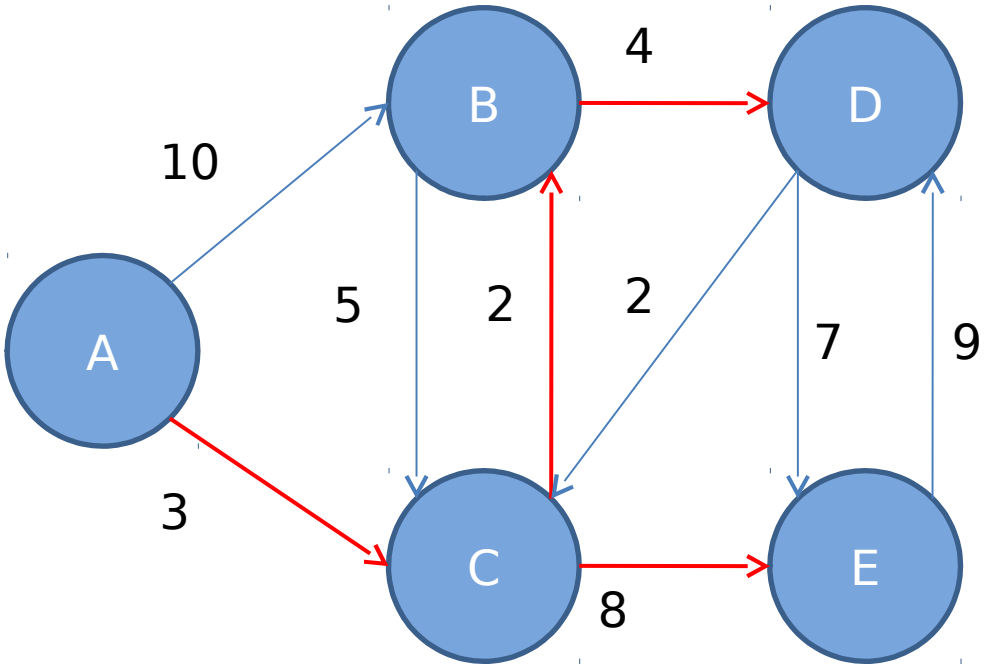
Cycle formed (B, C, D)

A => B, C, D = 10 – 2 = 8

A => B, C, D = 3 – 2 = 1

E => B, C, D = 9 – 4 = 5

Chu-Liu/Edmonds Algorithm



No cycle formed, we terminate

Chu-Liu/Edmonds Algorithm

- **Using this algorithm may unloop a cycle and create another cycle…**
- **… But since the length of tree is increasing, there will have an end eventually**
- **Worst Case is doing n-1 times**
- **Time Complexity is O(EV)**

Flow

- **A graph with weight (capacity)**
- **Mission: Find out the maximum flow from source to destination**
- **How?**

Edmond-Karp

- **Do {**
  - **Do BFS on the graph to find maximum flow in the graph**
  - **Add it to the total flow**
  - **For each edge in the maximum flow**
  - **Deduct the flow just passed**
- **} While the flow is not zero;**

The End